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The Louisiana State University and Agricultural and Mechanical Col., 1994

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# STRUCTURAL RESULTS FOR MATROIDS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

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# ABSTRACT

This dissertation solves some problems involving the structure of matroids. In Chapter 2, the class of binary matroids with no minors isomorphic to the prism graph, its dual, and the binary affine cube is completely determined. This class contains the infinite family of matroids obtained by sticking together a wheel and the Fano matroid across a triangle, and then deleting an edge of the triangle. In Chapter 3, we extend a graph result by D.W. Hall to matroids. Hall proved that if a simple, 3-connected graph has a $K_5$-minor, then it must also have a $K_{3,3}$-minor, the only exception being $K_5$ itself. We prove that if a 3-connected, binary matroid has an $M(K_5)$-minor, then it must also have a minor isomorphic to $M(K_{3,3})$ or its dual, the only exceptions being $M(K_5)$, a highly symmetric 12-element matroid $T_{12}$, and $T_{12}$ with any edge contracted. Chapter 4 consists of a collection of results on the intersection of circuits and cocircuits in binary matroids. In Chapter 5, we describe, in terms of excluded minors, the class of non-binary matroids with the property that a matroid is in the class if its restriction to every hyperplane is binary.

# CHAPTER I

# INTRODUCTION

## 1.1. Matroid structure

Several results in graph theory and matroid theory are of the type "for any matroid $M$, either $M$ has a certain structure, or $M$ contains certain minors, and not both". Results of this type can be divided naturally into two classes, depending on whether they are motivated by the structure imposed, or by the minors excluded. In the first case, given a class of matroids with a certain structure, we ask whether the class can be characterized in terms of excluded minors. Examples of such "structure-driven" results are stated below. The first, by Wagner (1937), extends Kuratowski's famous characterization of planar graphs (1930). The next two results are by Tutte (1958, 1959).

$K_5$ denotes the complete graph on five vertices, and $K_{3,3}$ denotes the complete bipartite graph with three vertices in each class. $U_{2,4}$ denotes the 4-point line. $F_7$ denotes the Fano matroid, and $F_7^*$ denotes its dual.

**Theorem 1.1.1.** *A graph is planar if and only if it has no minor isomorphic to $K_5$ or $K_{3,3}$.*

**Theorem 1.1.2.** *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

**Theorem 1.1.3.** *A matroid is regular if and only if it has no minor isomorphic to $U_{2,4}$, $F_7$, or $F_7^*$.*

The oppositely motivated problem is concerned with selecting certain matroids and examining the class of matroids with no minors isomorphic to them. Examples of such results by Hall (1943), Robertson and Seymour (1984), Brylawski (1971), and Oxley (1987a), respectively, are stated below.

For $r \geq 3$, $W_r$ denotes the wheel with $r$ spokes. $K_5\backslash e$ denotes the graph $K_5$ with an edge deleted, and $(K_5\backslash e)^*$ denotes the dual of $K_5\backslash e$.

**Theorem 1.1.4.** *Let $G$ be a 3-connected graph. $G$ has no minor isomorphic to $K_{3,3}$ if and only if $G$ is planar or $|V(G)| \leq 5$.*

**Theorem 1.1.5.** *Let $G$ be a simple, 3-connected graph with $|V(G)| \geq 4$. $G$ has no minor isomorphic to $K_5\backslash e$ if and only if $G$ is isomorphic to $K_{3,3}$, $(K_5\backslash e)^*$, or $W_r$, for some $r \geq 3$.*

**Theorem 1.1.6.** *$M$ is a binary matroid with no minor isomorphic to $M(W_3)$ if and only if $M$ is a series-parallel network.*

**Theorem 1.1.7.** *Let $M$ be a 3-connected binary matroid with $|E(M)| \geq 4$. Then $M$ has no minor isomorphic to $M(W_4)$ if and only if $M$ is isomorphic to $M(W_3)$, $F_7$, $F_7^*$, $M(Z_r)$, $M^*(Z_r)$, $M(Z_r)\backslash c_r$, or $M(Z_r)\backslash b_r$, for some $r \geq 4$, where $M(Z_r)$ is the matroid represented by the following matrix over $GF(2)$:*

$$
Z_r = \begin{array}{c}
\begin{array}{ccccccccc} a_1 & \cdots & a_r & b_1 & b_2 & b_3 & & b_r & c_r \end{array} \\
\left( \begin{array}{ccc|cccccc}
 & & & 0 & 1 & 1 & \cdots & 1 & 1 \\
 & & & 1 & 0 & 1 & \cdots & 1 & 1 \\
 & I_r & & 1 & 1 & 0 & \cdots & 1 & 1 \\
 & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
 & & & 1 & 1 & 1 & \cdots & 0 & 1
\end{array} \right)
\end{array}
$$

In Chapter 2 we present an excluded-minor result. The matroids which are excluded are $K_5\backslash e$, its dual $(K_5\backslash e)^*$, and the binary affine cube $AG(3,2)$. The binary matroids with no minors isomorphic to the above matroids are completely determined. In general, characterizing an excluded-minor class of matroids depends largely on making a suitable selection of minors to be excluded. The excluded-minor class obtained should not be too large, in which case it may be impossible to determine precisely, or too small, in which case it would be trivial. One hopes that the class contains infinite families of matroids which can be simply described. The class characterized in Chapter 2 contains the infinite family of matroids obtained by sticking together the Fano matroid and a wheel across a 3-point line.

In Chapter 3, we extend Hall's graph result, Theorem 1.1.4, which states that if $G$ is a 3-connected, simple graph and it has a minor isomorphic to $K_5$, then it must also have a minor isomorphic to $K_{3,3}$, the only exception being $K_5$ itself. A common way of obtaining results in matroid theory is to extend well established results in graph theory. While generalization as an end in itself is undesirable, concise matroid results that come from graphs, besides describing the structure of matroids, yield useful insights into the structure of graphs. The main result of Chapter 3 states that if $M$ is a 3-connected binary matroid and it has a minor isomorphic to $M(K_5)$, then it must also have a minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$, the only exceptions being $M(K_5)$, a highly symmetric 12-element matroid called $T_{12}$, and any single-element contraction of $T_{12}$.

Chapter 4 consists of a collection of results obtained while working on a conjecture in Chapter 14 of Oxley's book, *Matroid Theory* (1992, 14.8.3): Call a $k$-element set which is the intersection of a circuit and a cocircuit a *special $k$-set*. It is conjectured that if a binary matroid has a special $k$-set, then it must have a special $(k-2)$-set. We prove this conjecture for graphic matroids. In addition, we prove that if a binary matroid has a special $k$-set, for some $k \geq 4$, then it must have a special 6-set. The binary matroids with special 6-sets are determined and the regular matroids without special 6-sets are determined.

In Chapter 5 we prove a structure-driven result. For a class of matroids $\mathcal{M}$ closed under isomorphisms, direct sums, and the taking of minors, let $\mathcal{M}_1$ be the class of matroids with the property that a matroid is in $\mathcal{M}_1$ if its restriction to every hyperplane is in $\mathcal{M}$. Necessary and sufficient conditions are given for a matroid to be in $\mathcal{M}_1$. Several characterizations are given for $\mathcal{M}_1$ when $\mathcal{M}$ is binary. Finally, the Appendix contains the single-element extensions of various matroids, and other calculations delayed from earlier chapters.

### 1.2. Terminology and previous results

The matroid terminology used here will in general follow Oxley (1992). The *ground set*, *set of circuits*, *set of hyperplanes*, *rank*, and *corank* of the matroid $M$ are denoted by $E(M)$, $\mathcal{C}(M)$, $\mathcal{H}(M)$, $r(M)$, and $r^*(M)$, respectively. The *dual* of a matroid $M$ is denoted by $M^*$. We say a matroid is *self-dual* if it is isomorphic to its dual. A maximal independent set in $M$ is called a *basis*. A minimal dependent set in $M$ is called a *circuit*. A *cocircuit* is a circuit in $M^*$. A circuit with $t$ elements is called a *$t$-circuit*. A 3-element circuit is called a *triangle*. A single-element circuit is called a *loop*. A single-element cocircuit is called a *coloop*.

The *closure* of X, denoted by $cl(X)$, is the set $X \cup \{e : M$ has a circuit $C$ such that $e \in C \subseteq X \cup e\}$. If $X = cl(X)$, then $X$ is called a *flat* of $M$. A *hyperplane* of $M$ is a flat of rank $r(M)-1$. A subset $X$ of $E(M)$ is a *spanning* set of $M$ if $cl(X) = E(M)$.

The *cycle matroid* of a graph $G$ is denoted by $M(G)$. A matroid is said to be *graphic* if it can be represented by a graph. A matroid is said to be *cographic* if its dual is graphic. The *direct sum* of two matroids $M_1$ and $M_2$, denoted by $M_1 \oplus M_2$, is defined to be the matroid with ground set $E(M_1) \cup E(M_2)$ and set of circuits $\mathcal{C}(M_1) \cup \mathcal{C}(M_2)$.

If $A$ is a matrix with entries in a field $F$, then the matroid on the set of columns of $A$ that is induced by linear independence over $F$ is called the *vector matroid* $M(A)$ of $A$. The column vector corresponding to the $i$th column will be denoted by $\bar{i}$. A matroid is *binary* if it can be represented by a matrix over the field of two elements; that is, it is isomorphic to the vector matroid of a matrix over this field. A basic tool in this dissertation is the well-known fact that binary matroids are uniquely representable; that is, if $A$ and $A'$ are $r \times n$ matrices over $GF(2)$ such that the map which, for all $i \in \{1, 2, \ldots, n\}$, takes the $i$th column of $A$ to the $i$th column of $A'$ is an isomorphism from $M(A)$ to $M(A')$, then $A'$ can be transformed into $A$ by a sequence of operations, each of which consists of interchanging two rows or adding one row to another. If $A$ is an $r \times n$ matrix with column labels $1, 2, \ldots, n$, and $\bar{x}$ is a $1 \times r$ column vector, then denote by $A \cup \bar{x}$ the $r \times (n+1)$ matrix $A$ with the column $\bar{x}$ affixed at the end. Label the columns of $A \cup \bar{x}$ as $1, 2, \ldots, n, x$.

If $X$ is a subset of $E(M)$, then $r(X)$ denotes the rank of $X$ in $M$. The *deletion* of $X$ from $M$, denoted by $M\backslash X$, is the matroid with ground set $E(M) - X$, in which a subset $C$ of $E(M) - X$ is a circuit if and only if it is a circuit in $M$. The *contraction* of $X$ from $M$, denoted by $M/X$, is defined to be the matroid $(M^*\backslash X)^*$. If $M\backslash X = M/X$, then $X$ is called a *separator* of $M$. The matroid $M\backslash(E(M) - T)$ is denoted by $M|T$. A matroid $N$ is a *minor* of a matroid $M$ if $N \cong M\backslash X/Y$ for some disjoint subsets $X$ and $Y$ of

$E(M)$. The class of binary matroids with no minor isomorphic to the matroids $M_1, M_2, \ldots, M_k$ is denoted by $EX(M_1, M_2, \ldots, M_k)$.

An *automorphism* of a matroid $M$ is a permutation $\sigma$ of $E(M)$ such that $r(X) = r(\sigma(X))$ for all $X \subseteq E(M)$. The set of automorphisms of $M$ forms a group under composition. This automorphism group is *transitive* if, for every two elements $x$ and $y$ of $M$, there is an automorphism that maps $x$ to $y$.

A matroid $M$ is *3-connected* if it is connected and $E(M)$ cannot be partitioned into subsets $X$ and $Y$, each having at least two elements, such that $r(X) + r(Y) - r(M) = 1$. To eliminate trivial cases, we shall assume that a 3-connected matroid has at least four elements. It is routine to verify that $M^*$ is 3-connected if and only if $M$ is 3-connected. If $M$ and $N$ are matroids on the sets $S$ and $S \cup e$ where $e \notin S$, then $M$ is an *extension* of $N$ if $M \backslash e = N$, and $M$ is a *coextension* of $N$ if $M/e = N$. By duality, $M$ is a coextension of $N$ if and only if $M^*$ is an extension of $N^*$. If $N$ is a 3-connected matroid, then an extension $M$ of $N$ is 3-connected provided $e$ is not in a 1- or 2-element circuit of $N$ and $e$ is not a coloop of $N$. Likewise, $M$ is a 3-connected coextension of $N$ if $M^*$ is a 3-connected extension of $N^*$. If a matroid $M$ has $k$ single-element extensions, we denote them by $(M, ext1)$, $(M, ext2)$, $\ldots$, $(M, extk)$.

Let $M_1$ and $M_2$ be binary matroids such that $E(M_1) \cap E(M_2) = T$. If $T = \{p\}$, then $P(M_1, M_2)$ denotes the *parallel connection* of $M_1$ and $M_2$ with respect to $p$. If $M_1|T = M_2|T = \triangle$ say, where $\triangle$ is a triangle, then $P_\triangle(M_1, M_2)$ denotes the *generalized parallel connection* of $M_1$ and $M_2$ with respect to $\triangle$ (Oxley, 1992, 12.4). Seymour (1980) proved the following basic link between 3-connectedness and parallel connection:

**Theorem 1.2.1.** *A connected matroid $M$ is not 3-connected if and only if there are matroids $M_1$ and $M_2$, each of which has at least three elements, and is isomorphic to a minor of $M$, such that $M = P(M_1, M_2)\backslash p$, where $p$ is neither a loop nor a coloop of $M_1$ or $M_2$.* □

When $M$ decomposes as in this theorem, it is called the 2-*sum* of $M_1$ and $M_2$, and is denoted by $M_1 \oplus_2 M_2$. If $\{x, y\}$ is a circuit of the matroid $M$, we say that $x$ and $y$ are *in parallel* in $M$. If, instead, $\{x, y\}$ is a cocircuit of $M$, then we say that $x$ and $y$ are *in series* in $M$. If $N = M\backslash x$ and $x$ is in a 2-circuit then $M$ is called a *parallel extension* of $N$ and $N$ is called a *parallel deletion* of $M$. If $N = M/x$ and $x$ is in a 2-cocircuit then $M$ is called a *series extension* of $N$ and $N$ is called a *series contraction* of $M$. By duality, $M$ is a parallel extension of $N$ if and only if $M^*$ is a series extension of $N^*$. A matroid $M$ is said to be a *series-parallel extension* of a matroid $N$ if $M$ can be obtained from $N$ by a sequence of operations, each of which is either a series extension or a parallel extension. If $N$ is a single-element matroid, that is $N$ is an edge or a loop, then $M$ is called a *series-parallel network* (Oxley 1992, 5.4). A matroid $N$ is said to be a *parallel-minor* of $M$ if $N$ can be obtained from $M$ by a sequence of contractions and parallel deletions. A matroid $N$ is said to be a *series-minor* of $M$ if $N$ can be obtained from $M$ by a sequence of deletions and series-contractions. By duality, $N$ is a parallel minor of $M$ if and only if $N^*$ is a series minor of $M^*$.

Let $\mathcal{M}$ be a class of matroids that is closed under minors and under isomorphisms. A member $N$ of $\mathcal{M}$ is called a *splitter* for $\mathcal{M}$ if no 3-connected member of $\mathcal{M}$ has a proper minor isomorphic to $N$. We shall rely heavily on the following easy consequence of the Splitter Theorem (Seymour, 1980, 7.3):

**Theorem 1.2.2.** *Let $M$ and $N$ be 3-connected, binary matroids such that $N$ is a minor of $M$, $|E(N)| \geq 4$, and if $N \cong M(W_k)$, $M$ has no $M(W_{k+1})$-minor. Then there is a sequence $M_0, M_1, \ldots, M_n$ of 3-connected matroids such that $M_0 \cong N$, $M_n = M$ and, for all $i$ in $\{1, 2, \ldots, n\}$, $M_i$ is either an extension or a coextension of $M_{i-1}$.* □

A detailed explanation of the next two results may be found in Seymour (1980, 7.2, 14.2). Recall from Theorem 1.1.3 that a matroid is regular if it is binary and has no minor isomorphic to the Fano matroid or its dual. Let $R_{10}$ and $R_{12}$ be the vector matroids of the following matrices over $GF(2)$:

$$\begin{array}{cc} 1 \cdots 5 & \begin{array}{ccccc} 6 & 7 & 8 & 9 & 10 \end{array} \\ \left( I_5 \right. & \left| \begin{array}{ccccc} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array} \right) \end{array} \qquad \begin{array}{cc} 1 \cdots 6 & \begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \left( I_6 \right. & \left| \begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right) \end{array}$$

$R_{10}$ $\qquad$ $R_{12}$

**Figure 1.** $R_{10}$ and $R_{12}$

$R_{10}$ and $R_{12}$ are self-dual. Every single-element deletion of $R_{10}$ is isomorphic to the matroid $M(K_{3,3})$. Every single-element extension of $R_{10}$ is non-regular, and therefore we have the following proposition.

**Proposition 1.2.3.** *$R_{10}$ is a splitter for the class of regular matroids.*

In his famous decomposition theorem, Seymour proved that every regular matroid may be constructed by piecing together graphic and cographic matroids and copies of $R_{10}$.

**Theorem 1.2.4.** *Let $M$ be a 3-connected, regular matroid. Then either $M$ is graphic or cographic, or $M$ has a minor isomorphic to one of $R_{10}$ or $R_{12}$.* $\square$

In Chapters 2 and 4 we will use the following result by Dirac (1963). A simple proof for this result was given by Brown (1965) (see Bollobás 1976, p. 110). $K_{3,p}$ is the complete bipartite graph with three vertices in one class and $p$ vertices in the other class. $K'_{3,p}$, $K''_{3,p}$, and $K'''_{3,p}$ are the simple graphs obtained from $K_{3,p}$ by adding one, two, and three edges, respectively, joining vertices in the class containing three vertices. These graphs are shown in Figure 2.

**Theorem 1.2.6.** *$G$ is a simple, 3-connected graph without two vertex-disjoint cycles if and only if $G$ is isomorphic to $K_5$, $K_5\backslash e$, $K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$, or $K'''_{3,p}$, for some $p \geq 3$, or $W_r$, for some $r \geq 3$.* $\square$

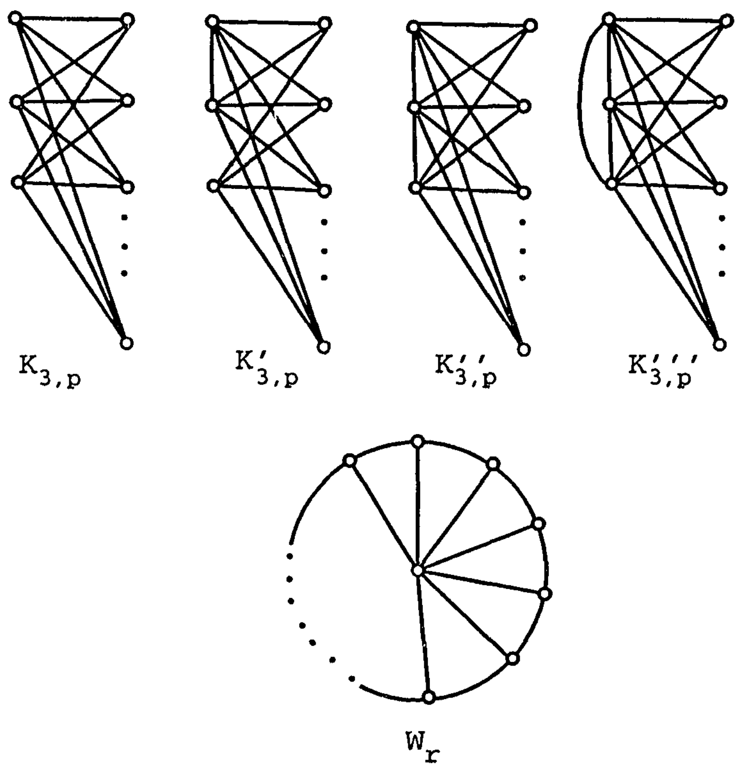
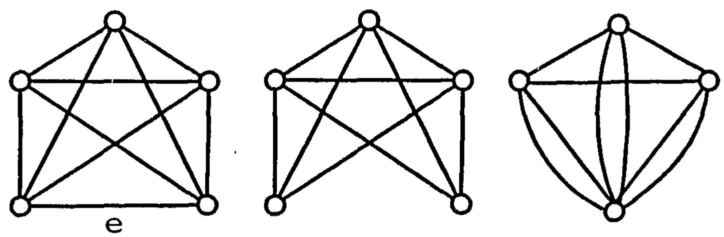



**Figure 2.** $K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$, $K'''_{3,p}$, and $W_r$

### 1.3. Single-element deletions and contractions

In this section we will discuss the operations of deletion and contraction in detail. Recall that a matroid $N$ is a minor of a matroid $M$ if $N \cong M\backslash X/Y$ for some disjoint subsets $X$ and $Y$ of $E(M)$. A minor of a graph $G$ is obtained from $G$ by deleting edges and vertices, and contracting edges. For example, the graphs that are obtained from $K_5$ by the deletion of a single edge and the contraction of a single edge are shown in Figure 3.




$K_5$ $K_5\backslash e$ $K_5/e$

**Figure 3.** $K_5$, $K_5\backslash e$, and $K_5/e$

Suppose $M$ is an $n$-element, rank-$r$ matroid representable over $GF(q)$. Then $M$ can be represented by a matrix $[I_r|D]$ over $GF(q)$, with columns labelled as $1, 2, \ldots, n$. A matrix in this form is said to be in *standard form.* The matrix $[-D^T|I_{n-r}]$ is a representation for $M^*$. Clearly, $M^*$ has $n$ elements and rank $n-r$. For example, matrix representations over $GF(2)$, for the matroid $P_9$ and its dual are shown below:

$$\begin{array}{c}
\begin{array}{cccc|ccccc} 1&2&3&4&5&6&7&8&9 \end{array} \\
\left(\begin{array}{cccc|ccccc}
 & & & & 0&1&1&1&1 \\
 & I_4 & & & 1&0&1&1&1 \\
 & & & & 1&1&0&1&0 \\
 & & & & 1&1&1&1&0
\end{array}\right) \\
P_9
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccc|ccccc} 1&2&3&4&5&6&7&8&9 \end{array} \\
\left(\begin{array}{cccc|ccccc}
0&1&1&1& & & & & \\
1&0&1&1& & & & & \\
1&1&0&1& & & I_5 & & \\
1&1&1&1& & & & & \\
1&1&0&0& & & & &
\end{array}\right) \\
P_9^*
\end{array}$$

When computing the single-element extensions of $M^*$, we may, without loss of generality, write the matrix representing $M^*$ in standard form and relabel the columns as $1, 2 \ldots, n$.

Before giving an algorithm to compute the single-element deletions and contractions of a matroid, we will describe the pivoting operation in detail. We assume all matrices have $n$ elements and rank $r$ and are in standard form $[I_r|D]$. In order to pivot on a non-zero element $[a_{i,j}]$, first divide each entry in row $i$ by $[a_{i,j}]$ to get a one in row $i$ and column $j$. Then convert column $j$ into the $i$th unit vector by the appropriate row operations. This completes the pivot.

Deleting elements in a matroid corresponds to deleting columns in the corresponding matrix. We will describe a method for obtaining the standard representation for a deletion. For $j$ in $\{r+1, r+2 \ldots, n\}$, deleting column $j$ simply involves removing that column from the matrix. For $j$ in $\{1, 2 \ldots, r\}$, first delete column $j$. Then if row $j$ consists of all zeros, delete it, otherwise find the first non-zero element, say $[a_{j,k}]$, and do a pivot on it to obtain the $j$th unit vector. Finally, write the matrix in standard form. For example, matrix representations for $P_9\backslash 2$ and $P_9\backslash 6$ are shown below:

$$\begin{array}{c} \begin{array}{cccccccc} 1 & 5 & 3 & 4 & 6 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ I_4 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 1 \end{array}\right) \\ P_9\backslash 2 \end{array} \qquad \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 0 \end{array}\right) \\ P_9\backslash 6 \end{array}$$

Next we will describe contractions and a method for obtaining a standard representation after a contraction. We may assume each column has at least one non-zero element, otherwise the column represents a loop and contraction is the same as deletion. To contract a column $j$ in $\{1, 2, \ldots, r\}$, simply delete row $j$ and column $j$ in $[I_r|D]$. To contract a column $j$ in $\{r+1, r+2, \ldots, n\}$, start by finding the first non-zero element in column $j$, say $[a_{i,j}]$. Do a pivot on this element to convert column $j$ into the $i$th unit vector. Then interchange column $j$ with column $i$. Finally, delete column $j$ and row $i$. For example, matrix representations for $P_9/2$ and $P_9/6$ are shown below:

$$
\begin{array}{ccc|ccccc}
1 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\qquad\qquad
\begin{array}{ccc|ccccc}
2 & 3 & 4 & 5 & 1 & 7 & 8 & 9
\end{array}
$$

$$
\left(\begin{array}{c|ccccc}
 & 0 & 1 & 1 & 1 & 1 \\
I_3 & 1 & 1 & 0 & 1 & 0 \\
 & 1 & 1 & 1 & 1 & 0
\end{array}\right)
\qquad\qquad
\left(\begin{array}{c|ccccc}
 & 1 & 0 & 1 & 1 & 1 \\
I_3 & 1 & 1 & 1 & 0 & 1 \\
 & 1 & 1 & 0 & 0 & 1
\end{array}\right)
$$

$$P_9/2 \qquad\qquad\qquad\qquad P_9/6$$

# CHAPTER II

# BINARY MATROIDS WITHOUT PRISMS, PRISM DUALS, AND CUBES

## 2.1. Motivation

The main result of this chapter describes the class of binary matroids with no minors isomorphic to $M(K_5\backslash e)$, its dual $M^*(K_5\backslash e)$, and the binary affine cube $AG(3,2)$ (see Figure 4). The graph $(K_5\backslash e)^*$ is known as the prism graph, hence the title of this chapter.

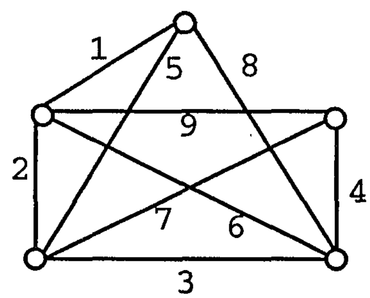
The motivation for the main theorem is Dirac's result, Theorem 1.2.6, in which the 3-connected, simple graphs without two vertex-disjoint cycles are determined. It is an easy consequence of Menger's theorem (1927) that excluding two vertex-disjoint cycles in a 3-connected simple graph is equivalent to excluding $(K_5\backslash e)^*$ as a minor. Thus, in the language of matroid theory, Dirac's result reads as follows:

**Theorem 2.1.1.** *$M$ is a 3-connected graphic matroid with no minor isomorphic to $M^*(K_5\backslash e)$ if and only if $M$ is isomorphic to $M(K_5)$, $M(K_5\backslash e)$, $M(K_{3,p})$, $M(K'_{3,p})$, $M(K''_{3,p})$, or $M(K'''_{3,p})$, for some $p \geq 3$, or $M(W_r)$, for some $r \geq 3$.* $\square$

In Theorem 1.1.5, Robertson and Seymour noted that the graphs without a $(K_5\backslash e)$-minor are precisely $(K_5\backslash e)^*$, $K_{3,3}$, and $W_r$, for some $r \geq 3$. This extends Dirac's result to the class of cographic matroids. This is because if $M$ is a cographic matroid with no $M^*(K_5\backslash e)$-minor, then $M^*$ is a graphic matroid with no $M(K_5\backslash e)$-minor. Therefore, since $W_r$ is self-dual, we have the following version of Theorem 1.1.5:

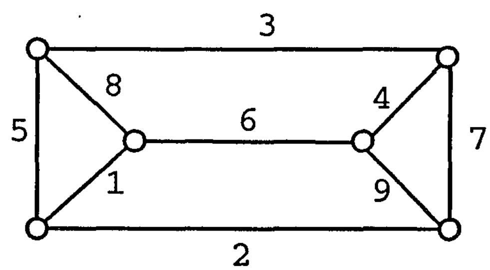
**Theorem 2.1.2.** *$M$ is a 3-connected cographic matroid with no minor isomorphic to $M^*(K_5\backslash e)$ if and only if $M$ is isomorphic to $M(K_5\backslash e)$, $M^*(K_{3,3})$, or $M(W_r)$, for some $r \geq 3$.* $\square$

$$\begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} & \begin{matrix} 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left( \begin{matrix} \\ & I_4 & \\ \\ \end{matrix} \right. & \left| \begin{matrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{matrix} \right) \end{array}$$



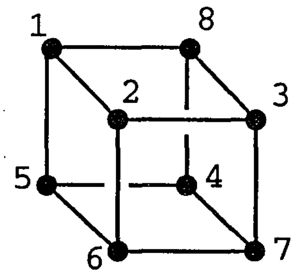

$M(K_5\backslash e)$

$$\begin{array}{cc} \begin{matrix} 5 & 6 & 7 & 8 & 9 \end{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \left( \begin{matrix} \\ \\ & I_5 & \\ \\ \\ \end{matrix} \right. & \left| \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{matrix} \right) \end{array}$$




$M^*(K_5\backslash e)$

$$\begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} & \begin{matrix} 5 & 6 & 7 & 8 \end{matrix} \\ \left( \begin{matrix} \\ & I_4 & \\ \\ \end{matrix} \right. & \left| \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix} \right) \end{array}$$




The 4-point planes are the six faces of the cube, the six diagonal planes, and two twisted planes: {1,3,4,6}, {2,5,7,8}.

$AG(3,2)$

**Figure 4.** $M(K_5\backslash e)$, $M^*(K_5\backslash e)$, and $AG(3,2)$.

Using Seymour's result on the decomposition of regular matroids, we can extend Dirac's result to the class of regular matroids. The only non-graphic, non-cographic member in the class is $R_{10}$.

**Proposition 2.1.3.** *$M$ is a 3-connected regular matroid with no minor iso-*
*$M^*(K_{3,3})$, $M(K_{3,p})$, $M(K'_{3,p})$, $M(K''_{3,p})$, or $M(K'''_{3,p})$, for some $p \geq 3$, $M(W_r)$,*
*$\backslash e)$,*
*for some $r \geq 3$, or $R_{10}$.*
*$M^*(K_{3,3})$, $M(K_{3,p})$, $M(K'_{3,p})$, $M(K''_{3,p})$, or $M(K'''_{3,p})$, for some $p \geq 3$, $M(W_r)$,*

*for some $r \geq 3$, or $R_{10}$.*

**Proof.** $R_{10}$ has no $M^*(K_5\backslash e)$-minor, since every single-element deletion of it is isomorphic to $M(K_{3,3})$. Theorems 2.1.1 and 2.1.2 imply that the remaining matroids have no $M^*(K_5\backslash e)$-minor. Conversely, let $M$ be a 3-connected, regular matroid with no minor isomorphic to $M^*(K_5\backslash e)$. If $M \cong R_{10}$, there is nothing to prove, so assume that $M \not\cong R_{10}$. Since $R_{10}$ is a splitter for the class of regular matroids, $M$ has no minor isomorphic to $R_{10}$. Theorem 1.2.4 implies that $M$ is graphic or cographic or has an $R_{12}$-minor. Figure 3 shows that $R_{12}$ has an $M^*(K_5\backslash e)$-minor, and therefore, we may assume that $M$ is graphic or cographic. The result follows from Theorems 2.1.1 and 2.1.2. $\square$

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 5 & 6 & 7 & 9 & 10 & 11 \end{array} \\ \left(\begin{array}{c|cccc} & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 0 \\ I_5 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 \end{array}\right) \end{array}$$

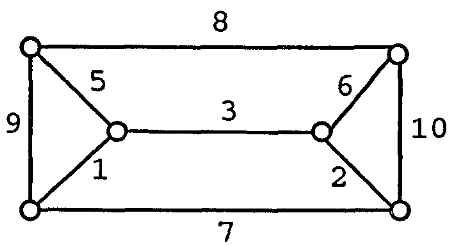
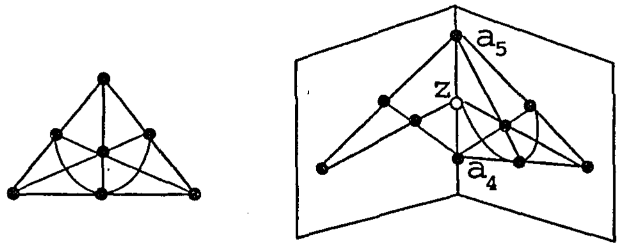



**Figure 5.** $R_{12}/4\backslash 8, 12 \cong M^*(K_5\backslash e)$

**Corollary 2.1.4.** *$M$ is a 3-connected, regular matroid with no minor isomorphic to $M^*(K_5\backslash e)$ or $M(K_5\backslash e)$ if and only if $M$ is isomorphic to $M(W_r)$, for some $r \geq 3$, $M(K_{3,3})$, $M^*(K_{3,3})$ or $R_{10}$.*




$F_7$ $\quad$ $P_\triangle(M(W_3),\ F_7)\backslash z$

**Figure 6.** $F_7$ and $P_\triangle(M(W_3),\ F_7)\backslash z$

One would like to extend Dirac's result further by characterizing the class of binary matroids with no $M^*(K_5\backslash e)$-minor, or the class of binary matroids with no $M^*(K_5\backslash e)$- and $M(K_5\backslash e)$-minors. Both of these problems appear difficult since the excluded-minor classes obtained are large. However, on excluding $M^*(K_5\backslash e)$, $M(K_5\backslash e)$ and $AG(3,2)$, one obtains an interesting class of matroids, all of whose members can be determined. This class is closed under duality since $AG(3,2)$ is self-dual. Furthermore, $AG(3,2)$ is the binary affine space of dimension 3, or rank 4, and is a fairly natural matroid to exclude.

The class contains the infinite family of 3-connected matroids of the form $P_\triangle(M(W_{r-1}),\ F_7) \setminus z$, for $r \geq 3$, where $P_\triangle(M(W_{r-1}),\ F_7)$ is obtained by sticking together $M(W_{r-1})$ and the Fano matroid $F_7$ across a triangle, and $z$ is the rim element of this triangle in $M(W_{r-1})$. The first member of this family, $P_\triangle(M(W_2),\ F_7)$, is precisely $F_7$. The next member of this family is $P_\triangle(M(W_3),\ F_7)\backslash z$. Geometric representations for both matroids are shown in Figure 6. In such a representation, a three-point line represents a triangle. The Fano matroid with any element deleted is a wheel with three spokes, which is in the class. The matroid $P_\triangle(M(W_3),\ F_7)\backslash z$ with either of $a_4$ or $a_5$ deleted is also a 3-connected matroid in the class. In general, the matroid $P_\triangle(M(W_3),\ F_7)\backslash z$ with any one of the remaining two elements of the triangle, $\triangle$ , deleted is also a 3-connected matroid in the class. In addition the class contains a certain 10-element, rank-5 matroid $M(E_5)$ which is a binary, non-regular, single-element extension of $M(K_{3,3})$.

## 2.2. The structure of the infinite family

We shall start with a description of the matrices that represent the matroids $P_\triangle(M(W_{r-1}),\ F_7) \setminus z$, over $GF(2)$. Familiarity with the structure of these matrices is crucial to the understanding of the proof. Furthermore, it appears that the class of binary matroids contains several similar infinite families. For an integer $r \geq 3$, let $\alpha_r$ be the $r \times r$ matrix obtained as follows:

(i) Begin with the $3 \times 3$ matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(ii) Adjoin to this matrix a row and a column of ones.

(iii) Then adjoin a row and a column each having its first two entries equal to one and its remaining entries equal to zero.

(iv) Repeat steps (ii) and (iii) alternately, beginning with (ii), until one obtains an $r \times r$ matrix. Thus, for example, $\alpha_4$, $\alpha_5$, and $\alpha_6$ are respectively:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let $\alpha'_r$ be the matrix obtained from $\alpha_r$ by deleting the last row and let $A_r = [I_r \ |\alpha'_{r+1}]$, the columns of this matrix being labelled, in order, as $b_1,\ b_2, \ldots b_r,\ a_1,\ a_2, \ldots, a_r,\ a_{r+1}$. Then $M(A_r)$ is the vector matroid of $A_{r(odd)}$ when $r$ is odd, and of $A_{r(even)}$ when $r$ is even, where $A_{r(odd)}$ and $A_{r(even)}$ are shown below:

$$A_{r(odd)} = \begin{array}{c} \begin{array}{ccc|cccccccccc} b_1 & \cdots & b_r & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \cdots & a_{r-1} & a_r & a_{r+1} \end{array} \\ \left(\begin{array}{c|cccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 \\ I_r & 1 & 1 & 1 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & \cdots & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right) \end{array}$$

$$A_{r(even)} = \begin{array}{c} \begin{array}{ccc|cccccccccc} b_1 & \cdots & b_r & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \cdots & a_{r-1} & a_r & a_{r+1} \end{array} \\ \left(\begin{array}{c|cccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 \\ I_r & 1 & 1 & 1 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & \cdots & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 1 & 0 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{array}\right) \end{array}$$

Observe that $M(A_3) \cong F_7$, $M(A_3)\backslash a_4 \cong M(W_3)$, and $M^*(A_r) \cong M(A_{r+1})\backslash\, a_{r+1}, a_{r+2}$, for all $r \geq 3$. In particular, $F_7^* \cong M(A_4)\backslash a_4, a_5$. Furthermore, for all $r \geq 3$, the matroid $M(A_r)\backslash a_{r+1}$ is self-dual. This is clear from the symmetry of the matrices.

The rest of this section is concerned with recognizing that $M(A_r)$ is isomorphic to $P_\triangle(M(W_{r-1}),\ F_7) \setminus z$. In order to show this, one needs to show that another representation for $M(A_r)$ is the $r \times (2r+1)$ matrix $U_r$ over $GF(2)$. Transforming the matrix $A_r$ to $U_r$ requires a long sequence of basis exchanges and therefore the details are tedious. In addition, the column labels in $U_{r(odd)}$ and $U_{r(even)}$ differ slightly so the two cases are dealt with separately. The matrix $U_{r(odd)}$ is shown below. In the matrix $U_{r(even)}$, the column labels $a_1$ and $b_2$ are swapped.

$$\begin{array}{cccccc|ccccccc} a_r & a_2 & a_{r+1} & a_{r-1} & \cdots & a_{r-2} & a_1 & b_2 & b_r & b_{r-2} & \cdots & b_{r-1} & b_1 \end{array}$$
$$\left(\begin{array}{c|ccccccc} & 1 & 0 & 0 & 0 & \cdots & 1 & 1 \\ & 1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ & 0 & 1 & 1 & 0 & \cdots & 0 & 1 \\ I_r & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right)$$

$U_{r(\text{odd})}$

We shall first show that $A_{r(odd)}$ can be transformed into the matrix $U_{r(odd)}$ by a sequence of the following operations: interchanging two columns or two rows, and adding one row to another. From the matrix $A_{r(odd)}$, observe that:

$$\begin{aligned} \bar{a}_1 &= \bar{a}_2 + \bar{a}_r && (1) \\ \bar{b}_1 &= \bar{a}_1 + \bar{a}_{r+1} && (2) \\ \bar{b}_1 &= \bar{b}_2 + \bar{a}_r && (3) \\ \bar{b}_2 &= \bar{a}_2 + \bar{a}_{r+1} && (4) \\ \bar{b}_3 &= \bar{a}_3 + \bar{a}_4 && (5) \\ \bar{b}_i &= \bar{a}_{i-1} + \bar{a}_{i+1} \text{ for } 4 \le i \le r && (6) \end{aligned}$$

Equations (3), (4), (5), and (6) imply that $\{a_2, a_3, \ldots, a_{r+1}\}$ forms a basis of $M(A_{r(odd)})$, since it has $r$ elements and spans $\{b_1, b_2, \ldots, b_r\}$. Equations (1), (2), and (3) imply that $\{a_1, a_2, a_r\}$, $\{b_1, a_1, a_{r+1}\}$, and $\{b_1, b_2, a_r\}$ are circuits and therefore, $b_1$ is in the plane spanned by $a_2$, $a_r$, and $a_{r+1}$. Next, we reorder the columns of $A_{r(odd)}$ so that the first $r$ columns, in order are

$\{a_r, a_2, a_{r+1}, a_{r-1}, a_{r-3}, \ldots, a_6, a_4, a_3, a_5, \ldots, a_{r-4}, a_{r-2}\}$ and the next $(r+1)$ columns, in order, are $\{a_1, b_2, b_r, b_{r-2}, b_{r-4}, \ldots, b_5, b_3, b_4, b_6, \ldots, b_{r-3}, b_{r-1}, b_1\}$. This matrix written in standard form is precisely $U_{r(odd)}$.

In the case when $r$ is even, $A_{r(even)}$ can be similarly transformed into $U_{r(even)}$. From the matrix $A_{r(even)}$, observe that:

$$\begin{aligned}
\bar{a}_1 &= \bar{a}_2 + \bar{a}_{r+1} && (1)\\
\bar{b}_1 &= \bar{a}_1 + \bar{a}_r && (2)\\
\bar{b}_1 &= \bar{b}_2 + \bar{a}_{r+1} && (3)\\
\bar{b}_2 &= \bar{a}_2 + \bar{a}_r && (4)\\
\bar{b}_3 &= \bar{a}_3 + \bar{a}_4 && (5)\\
\bar{b}_i &= \bar{a}_{i-1} + \bar{a}_{i+1} \text{ for } 4 \leq i \leq r && (6)
\end{aligned}$$

Once again equations (3), (4), (5), and (6) imply that $\{a_2, a_3, \ldots, a_{r+1}\}$ forms a basis of $M(A_{r(even)})$, since it has $r$ elements and spans $\{b_1, b_2, \ldots, b_r\}$. However, equations (1), (2), and (3) imply that $\{a_1, a_2, a_{r+1}\}$, $\{b_1, a_1, a_{r+1}\}$, and $\{b_1, b_2, a_{r+1}\}$ are circuits and therefore, $b_1$ is in the plane spanned by $a_2$, $a_r$, and $a_{r+1}$. Next, we reorder the columns of $A_{r(even)}$ so that the first $r$ columns, in order, are $\{a_r, a_2, a_{r+1}, a_{r-1}, a_{r-3}, \ldots, a_5, a_3, a_4, a_6, \ldots, a_{r-4}, a_{r-2}\}$ and the next $(r+1)$ columns, in order, are $\{b_2, a_1, b_r, b_{r-2}, b_{r-4}, \ldots, b_6, b_4, b_3, b_5, \ldots, b_{r-3}, b_{r-1}, b_1\}$. This matrix written in standard form is precisely $U_{r(even)}$. Hence, in any case $M(A_r) \cong M(U_r)$. Finally, observe that $M(U_r)\backslash b_1 \cong M(W_r)$ (see Figure 7).

Next, adjoin the column $(1, 0, 1, 0, \ldots, 0)^T$ to the matrix $U_r$ and label this column $z$. Let $\triangle = \{a_r, a_{r+1}, z\}$ .

**Lemma 2.2.1.** *$M(U_r) = P_\triangle(M(W_{r-1}),\ F_7) \setminus z$, where $M(W_{r-1})$ and $F_7$ are labelled as in Figure 10 and $z$ is the rim element of the triangle $\triangle$ in $M(W_{r-1})$.*

**Proof.** The matroid $M(U_r \cup \bar{z})/\Delta$, for which a matrix and a graphic representation are shown in Figure 8, has $S = \{a_1, a_2, b_1, b_2\}$ as a separator. This is because the removal of $S$ will disconnect the matroid. Thus, by a result of Brylawski (White, 1986, p. 186), $M(U_r \cup \bar{z}) = P_\Delta(M_1, M_2)$ where $M_1 = M(U_r \cup \bar{z})\backslash S$ and $M_2 = M(U_r \cup \bar{z})|(S \cup \Delta)$. It is clear from Figure 9 that $M(U_r \cup \bar{z})\backslash S \cong M(W_{r-1})$, where $\bar{z}$ is the rim element of the triangle $\Delta$, and that $M(U_r \cup \bar{z})|(S \cup \Delta) \cong F_7$. $\square$

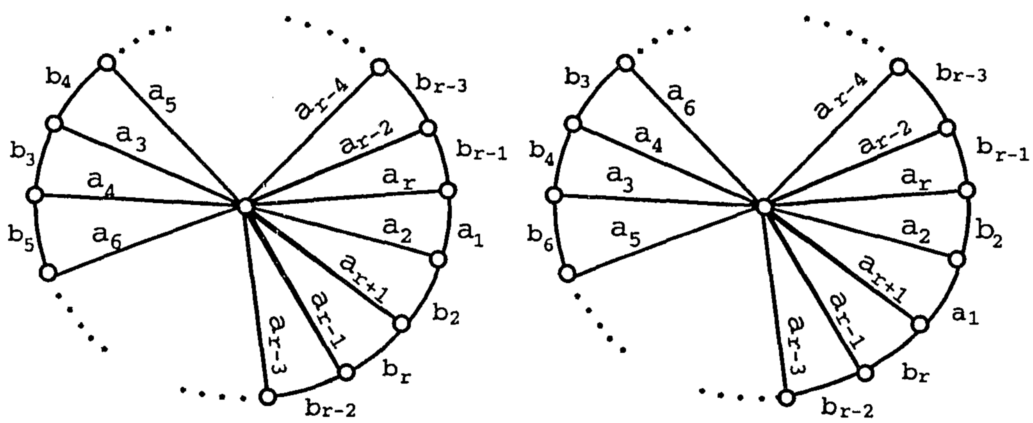
We can conclude from the previous lemma that $M(A_r) \cong P_\Delta(M(W_{r-1}),\ F_7) \setminus z$. From Figure 10, it is easy to see that:

$$M(A_r)\backslash a_r \cong M(A_r)\backslash a_{r+1}, \tag{2.2.2}$$

$$M(A_r)/b_i\backslash a_{i+1} \cong M(A_{r-1}) \text{ for all } i \in \{3, 4, \ldots, r\}, \tag{2.2.3}$$

$$M(A_r)/b_i\backslash a_{i-1} \cong M(A_{r-1}) \text{ for all } i \in \{4, 5 \ldots, r\}, \text{ and} \tag{2.2.4}$$

$$M(A_r)/b_3\backslash a_3 \cong M(A_{r-1}). \tag{2.2.5}$$




$U_{r(odd)}\backslash b_1$ $\qquad$ $U_{r(even)}\backslash b_1$

**Figure 7.** Graphic representations for $U_{r(odd)}\backslash b_1$ and $U_{r(even)}\backslash b_1$

$$\begin{array}{c}
\begin{array}{cccccc|cccccccc}
a_2 & a_{r-1} & \cdots & a_3 & \cdots & a_{r-2} & a_1 & b_2 & b_r & b_{r-2} & \cdots & b_{r-3} & b_{r-1} & b_1
\end{array} \\
\left(\begin{array}{c|cccccccc}
 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\
 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
I_{r-2} & 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0
\end{array}\right)
\end{array}$$

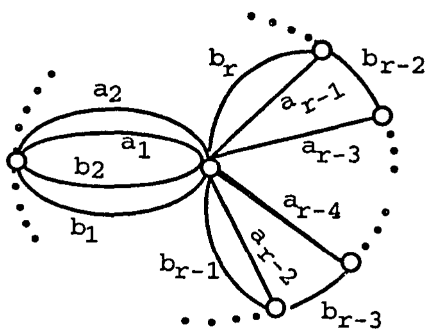



**Figure 8.** The matrix $(U_{r(odd)} \cup \bar{z})/\Delta$ and its graphic representation

$$(U_r \cup \bar{z})\backslash S = \begin{array}{c}
\begin{array}{ccccc|cccc}
a_r & a_{r+1} & a_{r-1} & \cdots & a_{r-2} & b_r & \cdots & b_{r-1} & z
\end{array} \\
\left(\begin{array}{ccccc|cccc}
1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 1 & \ldots & 0 & 1 \\
0 & 0 & 1 & \ldots & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0
\end{array}\right)
\end{array}$$

$$(U_r \cup \bar{z})|(S \cup \Delta) = \begin{array}{c}
\begin{array}{ccc|cccc}
a_r & a_2 & a_{r+1} & a_1 & b_2 & b_1 & z
\end{array} \\
\left(\begin{array}{c|cccc}
 & 1 & 0 & 1 & 1 \\
I_3 & 1 & 1 & 1 & 0 \\
 & 0 & 1 & 1 & 1
\end{array}\right)
\end{array}$$

**Figure 9.** $(U_{r(odd)} \cup \bar{z})\backslash S$ and $(U_{r(odd)} \cup \bar{z})|(S \cup \Delta)$

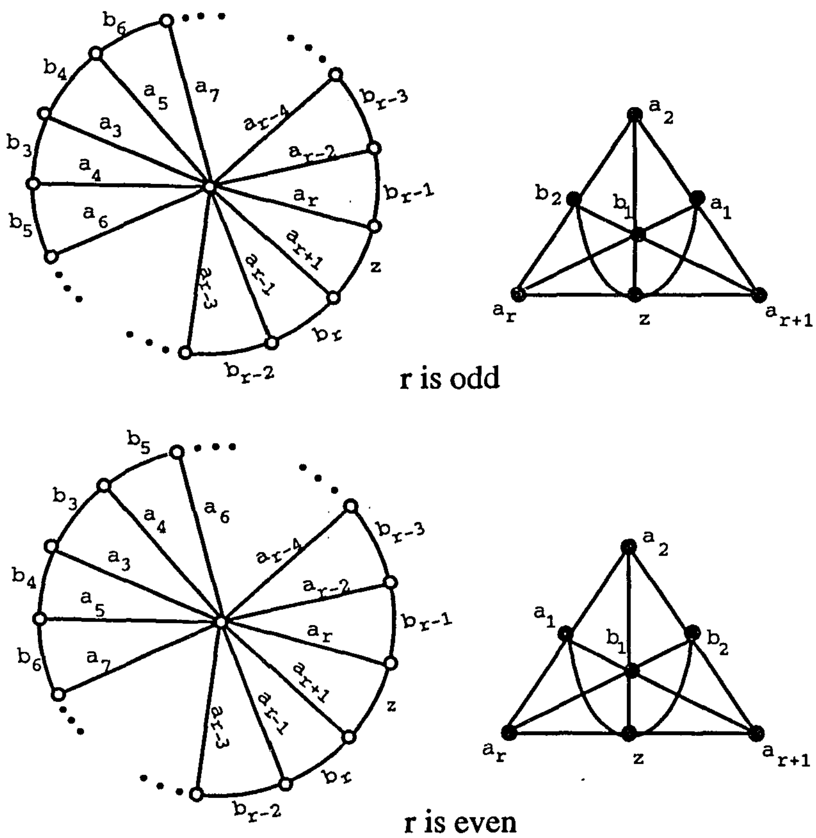



**Figure 10.** A graphic representation for $M(W_{r-1})$ and a geometric representation for $F_7$, in the case when $r$ is odd and even

Finally, consider the $5 \times 10$ matrix $E_5$ over $GF(2)$:

$$
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
E_5 = \left( \begin{array}{c|ccccc}
 & 0 & 1 & 1 & 1 & 1 \\
 & 1 & 0 & 1 & 1 & 0 \\
I_5 & 1 & 1 & 0 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & 0 \\
 & 1 & 1 & 0 & 0 & 0
\end{array} \right)
\end{array}
$$

$M(E_5)$ is self-dual. This is because the permutation of $\{1, 2 \ldots, 10\}$ that maps $(1, 2, \ldots, 10)$ to $(6, 8, 7, 9, 10, 1, 3, 2, 4, 5)$ is an isomorphism from $M(E_5)$ to $M^*(E_5)$. In addition, $M(E_5)\backslash 3 \cong M(K_{3,3})$.

**2.3. The main theorem.**

Let $\Theta$ denote the class of binary matroids with no minor isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, or $AG(3,2)$. The class $\Theta$ is closed under minors and duality. It follows from Theorem 1.2.1 that one can construct every matroid in $\Theta$ by beginning with the 3-connected members of $\Theta$ and repeatedly using the operations of direct sum and 2-sum. The following theorem is the main result of this chapter.

**Theorem 2.3.1.** *$M$ is a 3-connected binary matroid with no minor isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, or $AG(3,2)$ if and only if $M$ is isomorphic to $M(K_{3,3})$, $M^*(K_{3,3})$, $R_{10}$, $M(E_5)$, $M(W_r)$, $M(A_r)$, $M^*(A_r)$, or $M(A_r)\backslash a_{r+1}$, for some $r \geq 3$ .* $\square$

**Lemma 2.3.2.** *For all $r \geq 3$, $M(A_r)$ has no minor isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, or $AG(3,2)$.*

**Proof.** The proof is by induction on $r$. The lemma is true for $r = 3$ and $r = 4$. We will show that it is true for $M(A_5)$. From Figure 12 we see that, for $e \in \{a_1, a_2, b_1, b_2\}$, $M(A_5)\backslash e \cong M(W_5)$; for $e \in \{b_4, b_5\}$, $M(A_5)\backslash e$ is isomorphic to a series-parallel extension of $M(W_3)$; for $e \in \{a_3, a_4\}$, $M(A_5)\backslash e$ is isomorphic to a series extension of $P_9$; and $M(A_5)\backslash b_3$ is isomorphic to a series-parallel network. None of the above matroids have minors isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, or $AG(3,2)$. Therefore we need only consider $M(A_5)\backslash a_5$ and $M(A_5)\backslash a_6$.

Now (2.2.2) implies that $M(A_5)\backslash a_5 \cong M(A_5)\backslash a_6$. Furthermore, it is clear from the above information that, for $e \in \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5\}$, $M(A_5)\backslash a_6, e$ has no minor isomorphic to $M^*(K_5\backslash e)$ or $AG(3,2)$. Also note that $M(A_4)\backslash a_6, a_5 \cong M^*(A_5)$. Therefore $M(A_5)\backslash a_6$ has no minor isomorphic to $M^*(K_5\backslash e)$ or $AG(3,2)$. Since $M(A_5)\backslash a_6$ is self-dual it has no minor isomorphic to $M(K_5\backslash e)$. Therefore the lemma holds for $M(A_5)$.

Assume that the lemma holds for $M(A_{r-1})$ and consider $M(A_r)$, for some $r \geq 6$. Since $M(A_r) = P_\Delta(M(W_{r-1}),\ F_7) \setminus z$, it is easy to check from Figure 10 that, for all $e \in \{a_1, a_2, b_1, b_2\}$, $M(A_r)/e$ is isomorphic to a parallel extension of a wheel; $M(A_r)/a_r$ and $M(A_r)/a_{r+1}$ are isomorphic to series-parallel networks; and for all $e \in \{a_3, a_4, \ldots, a_{r-1}\}$, $M(A_r)/e$ is isomorphic to a series-parallel extension of $M(W_3)$. Observe that none of the above single-element contractions of $M(A_r)$ has a minor isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, or $AG(3,2)$. Finally, for all $e \in \{b_3, b_4, \ldots, b_r\}$, $M(A_r)/e$ is isomorphic to a parallel extension of $M(A_{r-1})$.

Now suppose that $M(A_r)$ has an $M^*(K_5\backslash e)$-, $M(K_5\backslash e)$- or $AG(3,2)$-minor. Then, for some $x \in E(M(A_r))$, the matroid $M(A_r)/x$ has such a minor. However, from the preceding information, this means that $M(A_{r-1})$ has such a minor; a contradiction to the induction hypothesis. $\square$

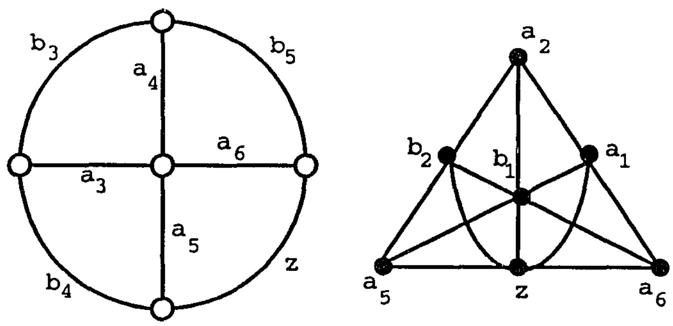



**Figure 12.** The matroid $M(A_5)$ with $z$

**Lemma 2.3.3** *$M(E_5)$ has no minor isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, or $AG(3,2)$.*

**Proof.** Since $M(E_5)$ is self-dual, it suffices to show that $M(E_5)$ has no minor isomorphic to $M(K_5\backslash e)$, or $AG(3,2)$. Suppose that $M(E_5)$ has an $M(K_5\backslash e)$- or $AG(3,2)$-minor. Then, for some $x \in E(M(E_5))$, the matroid $M(E_5)/x$ is isomorphic to $M(K_5\backslash e)$ or has an $AG(3,2)$-minor. However, $M(E_5)$ has precisely four non-isomorphic, single-element contractions, these being $M(A_4)$, $M^*(K_{3,3})$, a parallel extension of $F_7^*$, and a parallel extension of $M(W_4)$. None of these has an $M(K_5\backslash e)$- or $AG(3,2)$-minor; a contradiction. $\square$

**Proof of Theorem 2.3.1.** Corollary 2.1.4 implies that $M(K_{3,3})$, $M^*(K_{3,3})$, $R_{10}$, and $M(W_r)$ for any $r \geq 3$ are in $\Theta$. Lemma 2.3.2 implies that $M(A_r)$ for any $r \geq 3$ is in $\Theta$ and therefore $M^*(A_r)$ and $M(A_r)\backslash a_{r+1}$ for any $r \geq 3$ are also in $\Theta$. Lemma 2.3.3 implies that $M(E_5)$ is in $\Theta$.

Conversely, we shall prove that if $M$ is a 3-connected binary matroid in $\Theta$, then $M$ is one of the matroids listed in the statement of Theorem 2.3.1. If $M$ is regular, then clearly $M$ cannot have an $AG(3,2)$-minor and therefore, by Corollary 2.1.4, $M$ is isomorphic to $M(K_{3,3})$, $M^*(K_{3,3})$, $R_{10}$, or $M(W_r)$, for some $r \geq 3$,. Therefore, we may assume that $M$ is non-regular. Then, by Theorems 1.1.2 and 1.1.3, $M$ has a minor isomorphic to $F_7$ or $F_7^*$. Theorem 1.2.2 implies that there is a chain $M_0$, $M_1, \ldots, M_n$ of 3-connected matroids such that $M_0 \cong F_7$ or $F_7^*$, $M_n = M$, and, for all $i \in \{0, 1, \ldots, n-1\}$, $M_i$ is a single-element deletion or contraction of $M_{i+1}$. For the rest of the proof, we shall be concerned with the members of this chain. If $M \cong F_7$ or $F_7^*$, then there is nothing to show. Therefore, assume that this does not occur. We shall first suppose that $M_0 \cong F_7$. Then, since $F_7$ has no 3-connected, binary extension, $M_1$ is a 3-connected coextension of $F_7$, and, therefore, $M_1^*$ is a 3-connected extension of $F_7^*$. Seymour (1985) has noted that $F_7^*$ has precisely two nonisomorphic, binary, 3-connected, single-element extensions, these being $S_8$ and $AG(3,2)$ which are represented by the matrices $X_1$ and $X_2$ shown below:

$$X_1 = \left(\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \end{array}\right) \quad X_2 = \left(\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 0 \end{array}\right)$$

Since both $S_8$ and $AG(3,2)$ are self-dual, $M_1 \cong S_8 \cong M(A_4)\backslash a_5$.

Next, Oxley (1987) has noted that $S_8$ has precisely two nonisomorphic, binary, 3-connected, single-element extensions, these being $Z_4$ and $P_9$, represented by the matrices $X_3$ and $X_4$ shown below:

$$X_3 = \left(\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right) \quad X_4 = \left(\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

Observe that $Z_4$ has an $AG(3,2)$-minor, and $P_9 \cong M(A_4)$. Thus if $M_2$ is an extension of $M_1$, then $M_2 \cong M(A_4)$, and if $M_2$ is a coextension of $M_1$, then $M_2 \cong M^*(A_4)$. The next two lemmas determine the binary, 3-connected, single-element extensions of $P_9$ and $P_9^*$ in $\Theta$. The reader is referred to the Appendix for the detailed case analysis of the single-element extensions of $P_9$ and $P_9^*$.

**Lemma 2.3.4.** *$P_9$ has no 3-connected extension in $\Theta$.*

**Proof.** Table 1b in the Appendix implies that $P_9$ has three non-isomorphic, binary, 3-connected, single-element extensions, which are shown below:

$$\begin{array}{c} \begin{array}{cccc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \\ (P_9, ext1) \end{array} \qquad \begin{array}{c} \begin{array}{cccc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \\ (P_9, ext2) \end{array}$$

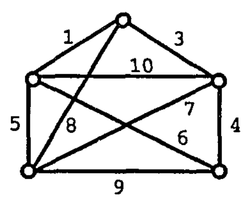
$$\begin{array}{c} \begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array} \\ \left(\begin{array}{c|cccccc} & 0&1&1&1&1&0\\ & 1&0&1&1&1&0\\ I_4 & 1&1&0&1&0&1\\ & 1&1&1&1&0&1 \end{array}\right) \end{array}$$

$(P_9, ext3)$

Clearly $(P_9, ext2)\backslash 8, 9 \cong AG(3, 2)$. Figure 13 shows that each of $(P_9, ext1)$ and $(P_9, ext3)$ has an $M(K_5\backslash e)$-minor.

$$\begin{array}{c} \begin{array}{ccccccccc} 1&5&3&4&6&7&8&9&10 \end{array} \\ \left(\begin{array}{c|ccccc} & 1&1&1&1&1\\ & 0&1&1&1&0\\ I_4 & 1&1&0&1&1\\ & 1&0&0&1&0 \end{array}\right) \end{array}$$

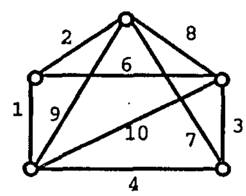
$(P_9, ext1)\backslash 2$




$M(K_5\backslash e)$

$$\begin{array}{c} \begin{array}{ccccccccc} 1&2&3&4&6&7&8&9&10 \end{array} \\ \left(\begin{array}{c|ccccc} & 1&1&1&1&0\\ & 0&1&1&1&0\\ I_4 & 1&0&1&0&1\\ & 1&1&1&0&1 \end{array}\right) \end{array}$$

$(P_9, ext3)\backslash 5$




$M(K_5\backslash e)$

**Figure 13.** $(P_9, ext1)\backslash 2 \cong M(K_5\backslash e)$ and $(P_9, ext3)\backslash 5 \cong M(K_5\backslash e)$

**Lemma 2.3.5.** *The only 3-connected, single-element extensions of $P_9^*$ in $\Theta$ are $M(A_5\backslash a_6)$ and $M(E_5)$.*

**Proof.** Table 2b in the Appendix implies that $P_9^*$ has precisely eight non-isomorphic, binary, 3-connected, single-element extensions which are shown below:

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&0\\ & 1&1&0&0&0 \end{array}\right)
\end{array}$$

$(P_9^*, ext1)$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&1\\ & 1&1&0&0&1 \end{array}\right)
\end{array}$$

$(P_9^*, ext2)$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&1\\ I_5 & 1&1&0&1&1\\ & 1&1&1&1&0\\ & 1&1&0&0&1 \end{array}\right)
\end{array}$$

$(P_9^*, ext3)$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&0\\ I_5 & 1&1&0&1&1\\ & 1&1&1&1&1\\ & 1&1&0&0&0 \end{array}\right)
\end{array}$$

$(P_9^*, ext4)$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&1\\ & 1&0&1&1&0\\ I_5 & 1&1&0&1&1\\ & 1&1&1&1&0\\ & 1&1&0&0&0 \end{array}\right)
\end{array}$$

$(P_9^*, ext5)$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&0\\ & 1&0&1&1&0\\ I_5 & 1&1&0&1&1\\ & 1&1&1&1&1\\ & 1&1&0&0&1 \end{array}\right)
\end{array}$$

$(P_9^*, ext6)$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&0\\ & 1&0&1&1&0\\ I_5 & 1&1&0&1&0\\ & 1&1&1&1&1\\ & 1&1&0&0&1 \end{array}\right)
\end{array}$$

$(P_9^*, ext7)$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\
\left(\begin{array}{c|ccccc} & 0&1&1&1&0\\ & 1&0&1&1&0\\ I_5 & 1&1&0&1&1\\ & 1&1&1&1&1\\ & 1&1&0&0&0 \end{array}\right)
\end{array}$$

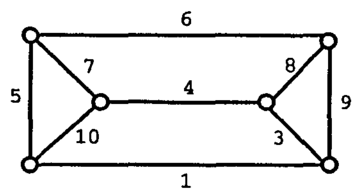
$(P_9^*, ext8)$

Each of $(P_9^*, ext1)$, $(P_9^*, ext2)$, $(P_9^*, ext3)$, $(P_9^*, ext4)$, and $(P_9^*, ext5)$ is self-dual, and $(P_9^*, ext8) \cong (P_9^*, ext6)^*$.

Clearly, $(P_9^*, ext2)/4\backslash 8 \cong AG(3,2)$ and $(P_9^*, ext3)/5\backslash 9 \cong AG(3,2)$. Figure 14 shows that each of $(P_9^*, ext4)$, $(P_9^*, ext6)$, and $(P_9^*, ext7)$ has an $M^*(K_5\backslash e)$-minor. Since $(P_9^*, ext6)$ has an $M^*(K_5\backslash e)$-minor, it follows by duality that $(P_9^*, ext8)$ has an $M(K_5\backslash e)$-minor. Notice that $(P_9^*, ext1) \cong M(A_5)\backslash a_6$ and $(P_9^*, ext5) \cong M(E_5)$. Lemmas 2.3.2 and 2.3.3, respectively, imply that $M(A_5)\backslash a_6$ and $M(E_5)$ are in $\Theta$. □

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 6 & 3 & 4 & 5 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{ccccc|cccc} &&&&& 1 & 1 & 1 & 1 \\ &&&&& 0 & 1 & 1 & 0 \\ &&I_5&&& 1 & 1 & 0 & 1 \\ &&&&& 1 & 0 & 0 & 1 \\ &&&&& 1 & 1 & 1 & 0 \end{array}\right) \end{array}$$

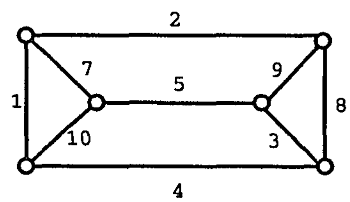
$(P_9^*, ext4)\backslash 2$




$(K_5\backslash e)^*$

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{ccccc|cccc} &&&&& 1 & 1 & 1 & 0 \\ &&&&& 0 & 1 & 1 & 0 \\ &&I_5&&& 1 & 0 & 1 & 1 \\ &&&&& 1 & 1 & 1 & 1 \\ &&&&& 1 & 0 & 0 & 1 \end{array}\right) \end{array}$$

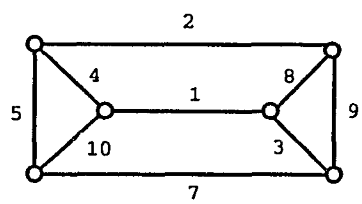
$(P_9^*, ext6)\backslash 6$




$(K_5\backslash e)^*$

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{ccccc|cccc} &&&&& 1 & 1 & 1 & 0 \\ &&&&& 0 & 1 & 1 & 0 \\ &&I_5&&& 1 & 0 & 1 & 0 \\ &&&&& 1 & 1 & 1 & 1 \\ &&&&& 1 & 0 & 0 & 1 \end{array}\right) \end{array}$$

$(P_9^*, ext7)\backslash 6$




$(K_5\backslash e)^*$

**Figure 14.** $(P_9, ext4)\backslash 2 \cong M^*(K_5\backslash e)$, $(P_9, ext6)\backslash 6 \cong M^*(K_5\backslash e)$, and $(P_9, ext7)\backslash 6 \cong M^*(K_5\backslash e)$

Lemma 2.3.4 implies that no single-element extension of $M(A_4)$ is in $\Theta$ while Lemma 2.3.5 implies that $M(A_5)\backslash a_6$ and $M(E_5)$ are the only 3-connected, single-element extensions of $M^*(A_4)$ in $\Theta$. Table 2b in the Appendix implies that $M(A_5)\backslash a_6$ is obtained from $P_9^*$ by adjoining either of the columns $(11000)^T$ or $(11111)^T$; and $M(E_5)$ is obtained by adjoining any one of the columns $(10100)^T$, $(01100)^T$, $(10011)^T$, and $(01011)^T$. Therefore, when we compute the extensions of $M(A_5)\backslash a_6$ and $M(E_5)$, we will only consider adjoining the above columns. Now $M_3$ can be either an extension of $M^*(A_4)$ or a coextension of $M(A_4)$. However, since $M(A_5)\backslash a_6$ and $M(E_5)$ are both self-dual, in either case, $M_3$ is isomorphic to $M(A_5)\backslash a_6$ or $M(E_5)$.

**Lemma 2.3.6.** *$M(E_5)$ is a splitter for $\Theta$. $M(A_5)$ is the only 3-connected single-element extension of $M(A_5)\backslash a_6$ in $\Theta$. Moreover, $M(A_5)$ has no 3-connected extension in $\Theta$.*

**Proof.** Table 5b in the Appendix implies that adjoining either of columns $(11000)^T$ or $(11111)^T$ to the matrix $E_5$, gives $(E_5, ext1)$. Adjoining either of columns $(01100)^T$ or $(01011)^T$, gives $(E_5, ext2)$. Finally, adjoining the column $(10011)^T$ gives $(E_5, ext7)$. Comparing Tables 2b and 5b we see that $(E_5, ext1)$ has a $(P_9^*, ext4)$-minor, which has an $M^*(K_5\backslash e)$-minor and $(E_5, ext2)$ has a $(P_9^*, ext8)$-minor, which has an $M(K_5\backslash e)$-minor. Figure 15 shows $(E_5, ext7)$ has an $M(K_5)\backslash e$-minor. Hence $M(E_5)$ is a splitter for $\Theta$.

Table 7b in the Appendix implies that adjoining column $(11111)^T$ to $A_5\backslash a_6$, gives $A_5$. Adjoining any one of columns $(10011)^T$, $(01011)^T$, $(10100)^T$, or $(01100)^T$ gives $(A_5\backslash a_6, ext7)$ which is shown in Figure 15. Clearly $(A_5\backslash a_6, ext7) \cong (E_5, ext1)$. Hence $A_5$ is the only 3-connected, single-element extension of $M(A_5)\backslash a_6$ in $\Theta$, and $M(A_5)$ has no 3-connected extension in $\Theta$.

$$\begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} & \begin{matrix} 6 & 7 & 8 & 9 & 10 & 11 \end{matrix} \end{array}$$

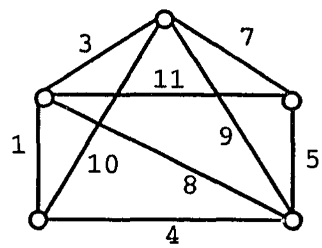
$$\left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

$(A_5 \backslash a_6, ext7)$

$$\begin{array}{cc} \begin{matrix} 1 & 3 & 4 & 5 \end{matrix} & \begin{matrix} 7 & 8 & 9 & 10 & 11 \end{matrix} \end{array}$$

$$\left(\begin{array}{c|ccccc} & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ I_4 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$(E_5, ext7)/2\backslash 6$




$(K_5 \backslash e)$

**Figure 15.** $(E_5, ext7)/2\backslash 6 \cong M(K_5\backslash e)$

**Lemma 2.3.7.** *For $r \geq 5$, the only columns that can be adjoined to the matrix representing $M^*(A_5)$ to give a representation of a 3-connected matroid in $\Theta$ are $(111111)^T$ and $(110000)^T$.*

**Proof.** Recall that $M^*(A_5) \cong M(A_6)\backslash a_6, a_7$, which is shown below:

$$\begin{array}{cc} \begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix} & \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \end{array}$$

$$\left(\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_6 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$M^*(A_5)$

From the above matrix, it is clear that

$$M^*(A_5)/b_6\backslash a_5 \cong M^*(A_4), \tag{1}$$
$$M^*(A_5)/b_4\backslash a_5 \cong M^*(A_4), \text{and} \tag{2}$$
$$M^*(A_5)/b_3\backslash a_3 \cong M^*(A_4). \tag{3}$$

Matrix representations for the above matroids are shown below:

$$\begin{array}{c} \begin{array}{ccccccccc} b_1 & b_2 & b_3 & b_4 & b_5 & a_1 & a_2 & a_3 & a_4 \end{array} \\ \left(\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 \end{array}\right) \\ M^*(A_5)/b_6\backslash a_5 \end{array} \qquad \begin{array}{c} \begin{array}{ccccccccc} b_1 & b_2 & b_3 & b_5 & b_6 & a_1 & a_2 & a_3 & a_4 \end{array} \\ \left(\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 \end{array}\right) \\ M^*(A_5)/b_4\backslash a_5 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccc} b_1 & b_2 & b_4 & b_5 & b_6 & a_1 & a_2 & a_4 & a_5 \end{array} \\ \left(\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 \end{array}\right) \\ M^*(A_5)/b_3\backslash a_3 \end{array}$$

Suppose that the column $\bar{x} = (x_1x_2x_3x_4x_5x_6)$ is adjoined to $A_6\backslash a_6, a_7$ to give a representation of a 3-connected matroid $M$ in $\Theta$. Then (1) implies that $M/b_6\backslash a_5, x \cong M^*(A_4)$. Therefore, $M/b_6\backslash a_5$ is a 3-connected, single-element extension, or a parallel extension of $M/b_6\backslash a_5, x$. Recall that the only columns that can be adjoined to the matrix representing $M^*(A_4)$ to get a matroid in $\Theta$ are $(11000)^T$, $(11111)^T$, $(10100)^T$, $(01100)^T$, $(10011)^T$, or $(01011)^T$. Therefore (1) implies that our choices for $\bar{x}$ are among the following columns:

$(1000001)^T$, $(010001)^T$, $(001001)^T$, $(000101)^T$, $(000011)^T$, $(011110)^T$, $(101110)^T$, $(110100)^T$, $(111100)^T$, $(1100000)^T$, $(110001)^T$, $(111110)^T$, $(111111)^T$, $(101000)^T$, $(101001)^T$, $(011000)^T$, $(011001)^T$, $(100110)^T$, $(100111)^T$, $(010110)^T$, and $(010111)^T$.

However, applying the same argument, (2) and (3) imply that our choices for $\bar{x}$ are among the following sets of columns, respectively:

$(100100)^T$, $(010100)^T$, $(001100)^T$, $(000110)^T$, $(000101)^T$, $(011011)^T$, $(101011)^T$, $(110001)^T$, $(111001)^T$, $(110000)^T$, $(110100)^T$, $(111011)^T$, $(111111)^T$, $(101000)^T$, $(101100)^T$, $(011000)^T$, $(011100)^T$, $(100011)^T$, $(100111)^T$, $(010011)^T$, and $(010111)^T$,

and

$(101000)^T$, $(011000)^T$, $(001100)^T$, $(001010)^T$, $(001001)^T$, $(010111)^T$, $(100111)^T$, $(110101)^T$, $(111001)^T$, $(110000)^T$, $(111000)^T$, $(110111)^T$, $(111111)^T$, $(100100)^T$, $(101100)^T$, $(010100)^T$, $(011100)^T$, $(100011)^T$, $(101011)^T$, $(010011)^T$, and $(011011)^T$.

The only columns in common among the above three sets of columns are $(110000)^T$, $(111111)^T$, $(100111)^T$, $(010111)^T$, and $(011000)^T$. Let $\bar{x} = (100111)^T$. The matroid $M$ represented by $A_6\backslash a_6, a_7 \cup \bar{x}$ is shown below:

$$A_6\backslash a_6, a_7 \cup \bar{x} = \begin{array}{c} \begin{array}{cccccccccccc} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & a_1 & a_2 & a_3 & a_4 & a_5 & x \end{array} \\ \left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ I_6 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array}$$

Figure 16 shows that $M$ has an $M^*(K_5\backslash e)$-minor. Similarly if $\bar{x} = (010111)^T$ then $M/b_2\backslash a_1, a_3 \cong M^*(K_5\backslash e)$ and if $\bar{x} = (011000)^T$ then $M/b_5\backslash a_2, a_3 \cong M^*(K_5\backslash e)$. Hence $(110000)^T$ and $(111111)^T$ are the only columns that can be adjoined to the matrix representing $M^*(A_5)$ to give a representation of a 3-connected matroid in $\Theta$.

$$
\begin{array}{c}
\begin{array}{ccccc|cccc}
b_2 & b_3 & b_4 & b_5 & b_6 & a_1 & a_4 & a_5 & x
\end{array} \\
\left(\begin{array}{c|cccc}
 & 1 & 1 & 1 & 0 \\
 & 1 & 1 & 0 & 0 \\
I_5 & 1 & 1 & 0 & 1 \\
 & 1 & 0 & 0 & 1 \\
 & 1 & 1 & 1 & 1
\end{array}\right)
\end{array}
$$

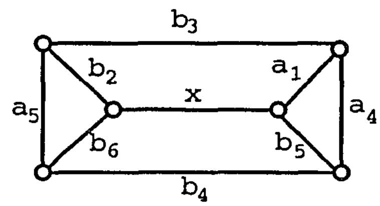



**Figure 16.** $M/b_1\backslash a_2, a_3 \cong M^*(K_5\backslash e)$

**Lemma 2.3.8.** *For $r \geq 5$, the only columns that can be adjoined to the matrix representing $M^*(A_r)$ to give a representation of a 3-connected matroid in $\Theta$ are $(111\ldots1)^T$ and $(110\ldots0)^T$.*

**Proof.** The proof is by induction on $r$. Lemma 2.3.7 implies that the result holds for $M^*(A_5)$. Assume that the result is true for $M^*(A_{r-1})$. Since the pattern differs for odd and even $r$, the two cases are dealt with separately. Recall that $M^*(A_r) \cong M(A_{r+1})\backslash a_{r+1}, a_{r+2}$. The matrices $A_{r+1}\backslash a_{r+1}, a_{r+2}$ in the cases when $r$ is odd and when $r$ is even, are shown below:

$$
\begin{array}{c}
\begin{array}{ccc|ccccccccc}
b_1 & \cdots & b_{r+1} & a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{r-2} & a_{r-1} & a_r
\end{array} \\
\left(\begin{array}{c|ccccccccc}
 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
 & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\
 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
I_{r+1} & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\
 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{array}\right)
\end{array}
$$

$M^*(A_r)$ when $r$ is odd

$$
\begin{array}{c}
\begin{array}{ccc|ccccccccc}
b_1 & \cdots & b_{r+1} & a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{r-2} & a_{r-1} & a_r
\end{array} \\
\left(\begin{array}{c|ccccccccc}
 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
 & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\
 & 1 & 1 & 1 & 1 & 0 & \cdots & 1 & 0 & 1 \\
 & 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\
I_{r+1} & 1 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\
 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

$M^*(A_r)$ when $r$ is even

From the above matrices it is clear that contracting row $b_{r+1}$ and deleting column $a_r$ gives $A_r\backslash a_r, a_{r+1}$. In addition, contracting row $b_{r-1}$ and deleting column $a_r$ also gives $A_r\backslash a_r, a_{r+1}$. Therefore we have the following relations:

$$M^*(A_r)/b_{r+1}\backslash a_r \cong M^*(A_{r-1}), \text{and} \tag{1}$$

$$M^*(A_r)/b_{r-1}\backslash a_r \cong M^*(A_{r-1}). \tag{2}$$

Let $r$ be odd. Suppose that the column $\bar{x} = (x_1 x_2 \dots x_{r+1})^T$ is adjoined to $A_{r+1}\backslash a_{r+1}, a_{r+2}$ to give a representation of a 3-connected matroid $M$ in $\Theta$. Then (1) implies that $M/b_{r+1}\backslash a_r, x \cong M^*(A_{r-1})$. Therefore, $M/b_{r+1}\backslash a_r$ is a 3-connected, single-element extension or a parallel extension of $M/b_{r+1}\backslash a_r, x$. Applying the induction hypothesis to $M/b_{r+1}\backslash a_r, x$, we see that our choices for $\bar{x}$ are among $(110\dots00)^T$, $(110\dots01)^T$, $(11\dots10)^T$, $(11\dots11)$, $b'_1, \dots, b'_r$, and $a'_1, \dots, a'_{r-1}$, where:

$b'_i$ is the column $b_i$ with 1 in the $(r+1)$st position, for all $i \in \{1, 2, \dots, r\}$; and

$a'_j$ is the column $a_j$ with 0 in the $(r+1)$st position, for all $j \in \{1, \dots, r-1\}$.

However, (2) implies that $M/b_{r-1}\backslash a_r, x \cong M^*(A_{r-1})$, so applying the induction hypothesis to $M/b_{r-1}\backslash a_r, x$, we see that our choices for $\bar{x}$ are among

$(110\ldots0000)^T$, $(110\ldots0100)^T$, $(11\ldots1011)^T$, $(11\ldots1111)^T$, $b''_1$, …, $b''_{r-2}$, $b''_r$, $b''_{r+1}$, and $a''_1$, …, $a''_{r-1}$, where:

$b''_i$ is the column $b_i$ with 1 in the $(r-1)$st position, for all $i \in \{1,\ldots,r-2,r,r+1\}$;

and $a''_j$ is the column $a_j$ with 0 in the $(r-1)$st position, for all $j \in \{1,2\ldots,r-1\}$. Observe that the columns occuring in both of the above sets of columns are $(110\ldots00)^T$, $(11\ldots11)^T$, and $(00\ldots0101)^T$. Suppose $\bar{x} = (00\ldots0101)^T$. The matrix $A_{r+1}\backslash a_{r+1}, a_{r+2} \cup \bar{x}$ representing $M$ is shown below:

$$
\begin{array}{ccc|cccccccccc}
b_1 & \cdots & b_{r+1} & a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{r-2} & a_{r-1} & a_r & x \\
\hline
 & & & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
 & & & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\
 & & & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
 & I_{r+1} & & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
 & & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
 & & & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\
 & & & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1
\end{array}
$$

$A_{r+1}\backslash a_{r+1}, a_{r+2} \cup \bar{x}$

Figure 17 shows that $M$ has an $M^*(K_5\backslash e)$-minor. Therefore $\bar{x}$ is one of $(110\ldots0)^T$ or $(11\ldots1)^T$.

$$
\begin{array}{ccccc|cccc}
b_1 & b_2 & b_3 & b_{r-1} & b_{r+1} & a_2 & a_4 & a_r & x \\
\hline
 & & & & & 1 & 1 & 1 & 0 \\
 & & & & & 0 & 1 & 1 & 0 \\
 & & I_5 & & & 1 & 1 & 0 & 0 \\
 & & & & & 1 & 1 & 0 & 1 \\
 & & & & & 1 & 1 & 1 & 1
\end{array}
$$

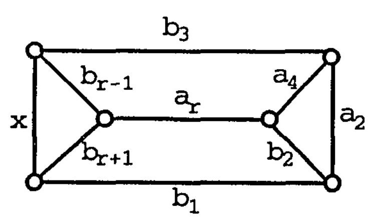



**Figure 17.** $M/\{b_4, b_5, \ldots, b_{r-2}, b_r\}\backslash\{a_1, a_3, a_5, a_6\ldots, a_{r-1}\} \cong M^*(K_5\backslash e)$

The case when $r$ is even is similar. Suppose the column $\bar{x} = (x_1 x_2 \dots x_{r+1})^T$ is adjoined to $A_{r+1} \backslash a_{r+1}, a_{r+2}$ to give a representation of a 3-connected matroid $M$ in $\Theta$. Then (1) implies that $M/b_{r+1} \backslash a_r, x \cong M^*(A_{r-1})$. Therefore, once again, $M/b_{r+1} \backslash a_r$ is a 3-connected, single-element extension or a parallel extension of $M/b_{r+1} \backslash a_r, x$. Applying the induction hypothesis to $M/b_{r+1} \backslash a_r, x$, we see that our choices for $\bar{x}$ are among $(110\dots00)^T$, $(110\dots01)^T$, $(11\dots10)^T$, $(11\dots11)$, $b'_1$, …, $b'_r$, and $a'_1$, …, $a'_{r-1}$, where:

$b'_i$ is the column $b_i$ with 1 in the $(r+1)$st position, for all $i \in \{1, 2, \dots, r\}$; and $a'_j$ is the column $a_j$ with 1 in the $(r+1)$st position, for all $j \in \{3, 4, \dots, r-1\}$; and $a'_j$ is the column $a_j$ with 0 in the $(r+1)$st position, for all $j \in \{1, 2\}$.

However, (2) implies that $M/b_{r-1} \backslash a_r, x \cong M^*(A_{r-1})$, so applying the induction hypothesis to $M/b_{r-1} \backslash a_r, x$, we see that our choices for $\bar{x}$ are among $(110\dots00)^T$, $(110\dots01)^T$, $(11\dots10)^T$, $(11\dots11)^T$, $b''_1$, …, $b''_{r-2}$, $b''_r$, $b''_{r+1}$, and $a''_1$, …, $a''_{r-1}$, where:

$b''_i$ is the column $b_i$ with 1 in the $(r-1)$st position, for all $i \in \{1, \dots, r-2, r, r+1\}$; and $a''_j$ is the column $a_j$ with 1 in the $(r-1)$st position, for all $j \in \{3, 4 \dots, r-1\}$; and $a''_j$ is the column $a_j$ with 0 in the $(r-1)$st position, for all $j \in \{1, 2\}$

Observe again that the columns occuring in both of the above sets of columns are $(110\dots00)^T$, $(11\dots11)^T$, and $(00\dots0101)^T$. Suppose $\bar{x} = (00\dots0101)^T$. The matrix $A_{r+1} \backslash a_{r+1}, a_{r+2} \cup \bar{x}$, when r is even, is a representation for $M$ and $M/\{b_3, b_4, b_6, b_7 \dots, b_{r-2}, b_r\} \backslash \{a_1, a_3, a_5, a_6 \dots, a_{r-1}\} \cong M^*(K_5 \backslash e)$. Therefore $\bar{x}$ is one of $(110\dots0)^T$ or $(11\dots1)^T$. □

**Corollary 2.3.9.** For $r \geq 5$, the unique 3-connected single-element extension of $M^*(A_r)$ in $\Theta$ is $M(A_{r+1}) \backslash a_{r+2}$. Moreover, the unique 3-connected single-element extension of $M(A_{r+1}) \backslash a_{r+2}$ in $\Theta$ is $M(A_{r+1})$.

**Proof.** (2.2.2) and the fact that $M^*(A_r) \cong M(A_{r+1})\backslash a_{r+1}, a_{r+2}$ imply that, on adjoining either of $(110\ldots0)^T$ or $(11\ldots1)^T$ to the matrix representing $M^*(A_r)$, we get a representation for $M(A_{r+1})\backslash a_{r+2}$. Therefore, Lemma 2.3.8 implies that $M(A_{r+1})\backslash a_{r+2}$ is the unique 3-connected single-element extension of $M^*(A_r)$ in $\Theta$. Moreover, adjoining both $(110\ldots0)^T$ and $(11\ldots1)^T$ gives $M(A_{r+1})$, which is the unique extension of $M(A_{r+1})\backslash a_{r+2}$ in $\Theta$. □

**Corollary 2.3.10.** *For $r \geq 5$, $M(A_r)$ has no 3-connected extension in $\Theta$.*

**Proof.** The proof follows from Corollary 2.3.9 and the fact that $M^*(A_r) \cong M(A_{r+1})\backslash\ a_{r+1}, a_{r+2}$, for all $r \geq 3$. □

Returning to the proof of Theorem 2.3.1, recall that $M_3$ is isomorphic to $M(E_5)$ or $M(A_5)\backslash a_6$. Therefore, $M_4$ is an extension or a coextension of $M(E_5)$ or $M(A_5)\backslash a_6$. Lemma 2.3.6 asserts that $M(E_5)$ is a splitter for $\Theta$. Therefore, $M_4$ is an extension or coextension of $M(A_5)\backslash a_6$. Since $M(A_5)\backslash a_6$ is self-dual, Lemma 2.3.6 implies that $M_4 \cong M(A_5)$ or $M^*(A_5)$. Corollaries 2.3.9 and 2.3.10 and the fact that $M(A_6)\backslash a_7$ is self-dual imply that $M_5 \cong M(A_6)\backslash a_7$.

To complete the proof it remains to prove that, for all $j \geq 2$, $M_{2j} \cong M(A_{j+3})$ or $M^*(A_{j+3})$ and $M_{2j+1} \cong M(A_{j+3})\backslash a_{j+2}$. The proof is by induction on $j$. The result is true for $j = 2$, since $M_4 \cong M(A_5)$ or $M^*(A_5)$ and $M_5 \cong M(A_6)\backslash a_7$. Assume the result holds for $k < j$ and let $k = j$. Then, by the induction hypothesis, $M_{2j-2} = M_{2(j-1)} \cong M(A_{j+2})$ or $M^*(A_{j+2})$ and $M_{2j-1} \cong M(A_{j+3})\backslash a_{j+4}$. Since $M(A_{j+3})\backslash a_{j+4}$ is self-dual, Corollary 2.3.9 implies that $M_{2j} \cong M(A_{j+3})$ or $M^*(A_{j+3})$. Finally, Corollaries 2.3.9 and 2.3.10 and duality imply that $M(A_{2j+1}) \cong M(A_{j+4})\backslash a_{j+5}$. □

It is well known that the binary, 3-connected, single-element extensions of the wheel with 4 spokes, $M(W_4)$, are $M(K_5\backslash e)$, $M^*(K_{3,3})$, and $P_9$. In his paper "Binary matroids with no 4-wheel minor", Oxley (1987) determined the binary matroids with no minors isomorphic to $P_9$ or $P_9^*$. Therefore, an interesting class of matroids would be those binary matroids with no minors isomorphic to $M(K_5\backslash e)$, $M^*(K_{3,3})$, or their duals. Using Theorem 2.3.1 we can determine an important subclass of the above class.

**Proposition 2.3.11.** *$M$ is a 3-connected, binary matroid with no minor isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, $M(K_{3,3})$, $M^*(K_{3,3})$, and $AG(3,2)$ if and only if $M$ is isomorphic to one of $M(W_r)$, $M(A_r)$, $M^*(A_r)$, or $M(A_r)\backslash a_{r+1}$, for some $r \geq 3$.*

**Proof.** The proof of Lemma 2.3.4 implies that $M(A_r)$ has no $M(K_{3,3})$-minor and each of $M(E_5)$ and $R_{10}$ has an $M(K_{3,3})$-minor. Therefore, the proposition follows from Theorem 2.3.1. □

In the next section we show that, in fact, there is not much difference between the class of binary matroids with no minors isomorphic to $M^*(K_5\backslash e)$, $M(K_5\backslash e)$, $M(K_{3,3})$, or $M^*(K_{3,3})$, and the class of binary matroids with no minors isomorphic to $M^*(K_5\backslash e)$ or $M(K_5\backslash e)$. This means that excluding $M(K_{3,3})$, or $M^*(K_{3,3})$ has no substantial effect on the answer.

## 2.4. The binary matroids with an $\mathbf{M(K_{3,3})}$-minor

In this section we will prove that a binary matroid with a minor isomorphic to $M(K_{3,3})$ must also have a minor isomorphic to $M(K_5\backslash e)$ or $M^*(K_5\backslash e)$, the only exceptions being $M(K_{3,3})$, $R_{10}$, and $M(E_5)$.

The following lemma was independently discovered and proved by Chula Jayavardane, a student of Neil Robertson at Ohio State University.

**Lemma 2.4.1.** *$R_{10}$ and $M(E_5)$ are splitters for the class of binary matroids with no minors isomorphic to $M^*(K_5\backslash e)$ or $M(K_5\backslash e)$.*

**Proof.** Comparing Tables 5b and 2b in the Appendix, it is clear that every extension of $M(E_5)$, except $(E_5, ext6)$ and $(E_5, ext7)$, has a minor isomorphic to $(P_9^*, ext4)$ or $(P_9^*, ext8)$. Lemma 2.3.5 implies that $(P_9^*, ext4)$ has an $M^*(K_5\backslash e)$-minor, and $(P_9^*, ext8)$ has an $M(K_5\backslash e)$-minor. Lemma 2.3.6 implies that $(E_5, ext7)$ has an $M(K_5\backslash e)$-minor. $(E_5, ext6)/5\backslash 2 \cong M(K_5\backslash e)$. Therefore, every extension of $M(E_5)$ has an $M^*(K_5\backslash e)$- or $M(K_5\backslash e)$-minor. Since $M(E_5)$ is self-dual, the coextensions of $M(E_5)$ are the duals of the extensions of $M(E_5)$.

Table 6b in the Appendix implies that $R_{10}$ has two nonisomorphic, binary, 3-connected, single-element extensions, for which matrix representations are shown below:

$$\begin{array}{cc} 1\ldots5 & \begin{array}{cccccc} 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \multicolumn{2}{c}{\left(\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 0 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right)} \end{array} \qquad \begin{array}{cc} 1\ldots5 & \begin{array}{cccccc} 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \multicolumn{2}{c}{\left(\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 0 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 1 & 1 \end{array}\right)} \end{array}$$

$(R_{10}, ext1)$ $\qquad$ $(R_{10}, ext2)$

$(R_{10}, ext1)$ is isomorphic to $(E_5, ext7)$, which has an $M(K_5\backslash e)$-minor. Figure 19 shows that $(R_{10}, ext2)$ has an $M(K_5\backslash e)$-minor. Again, since $R_{10}$ is self-dual, the coextensions of $R_{10}$ are the duals of the extensions of $R_{10}$.

$$\begin{array}{c} \begin{array}{cccc|ccccc} 1 & 2 & 3 & 4 & 6 & 7 & 8 & 9 & 11 \end{array} \\ \left(\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 0 & 1 \\ I_4 & 0 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 1 \end{array}\right) \end{array}$$

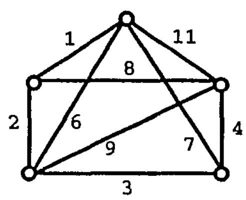



**Figure 18.** $(R_{10}, ext2)/5\backslash 10 \cong M(K_5\backslash e)$

**Proposition 2.4.2.** *Suppose $M$ is a 3-connected binary matroid with an $M(K_{3,3})$-minor. Then either $M$ has an $M(K_5\backslash e)$- or $M^*(K_5\backslash e)$- minor, or $M$ is isomorphic to $M(K_{3,3})$, $R_{10}$, or $M(E_5)$.*

**Proof.** $M(K_{3,3})$ has no minor isomorphic to $M(K_5\backslash e)$ or $M^*(K_5\backslash e)$. Theorem 2.3.1 implies that $R_{10}$ and $M(E_5)$ have no minor isomorphic to $M(K_5\backslash e)$ or $M^*(K_5\backslash e)$. Now suppose $M$ is a 3-connected binary matroid with an $M(K_{3,3})$-minor. Theorem 1.2.2 implies that there is a chain $M_0$, $M_1$, $\ldots$, $M_n$ of 3-connected matroids such that $M_0 \cong M(K_{3,3})$, $M_n = M$, and for all $i \in \{0, 1, \ldots, n-1\}$, $M_{i+1}$ is a single-element extension or coextension of $M_i$. For the rest of the proof, we will be concerned with the members of this chain. If $M \cong M(K_{3,3})$, then we have nothing to prove, therefore, assume this does not occur. Then $M_0 \cong M(K_{3,3})$ and $M_1$ is an extension or coextension of $M(K_{3,3})$.

Suppose that $M_1$ is a coextension of $M(K_{3,3})$. Note that the coextensions of $M(K_{3,3})$ are the duals of the extensions of $M^*(K_{3,3})$. Table 4b in the Appendix implies that $M^*(K_{3,3})$ has precisely one 3-connected single-element extension. Observe that $(K^*_{3,3}, ext1) \cong (P_9, ext1)$, which has a minor isomorphic to $M(K_5\backslash e)$. If $M_1 \cong (K^*_{3,3}, ext1)^*$, then it has an $M^*(K_5\backslash e)$-minor, and therefore so does $M$. So assume $M_1$ is an extension of $M(K_{3,3})$. Table 3b in the Appendix implies that $M(K_{3,3})$ has precisely four non-isomorphic, binary, 3-connected, single-element extensions, for which matrix representations are shown below.

$$\begin{array}{cccccc} 1\ldots5 & 6 & 7 & 8 & 9 & 10 \\ \end{array}$$
$$\left(\begin{array}{c|ccccc} & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$(K_{3,3}, ext1)$

$$\begin{array}{cccccc} 1\ldots5 & 6 & 7 & 8 & 9 & 10 \\ \end{array}$$
$$\left(\begin{array}{c|ccccc} & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ I_5 & 1 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$(K_{3,3}, ext2)$

$$\begin{array}{cccccc} 1\ldots5 & 6 & 7 & 8 & 9 & 10 \\ \end{array}$$
$$\left(\begin{array}{c|ccccc} & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$(K_{3,3}, ext3)$

$$\begin{array}{cccccc} 1\ldots5 & 6 & 7 & 8 & 9 & 10 \\ \end{array}$$
$$\left(\begin{array}{c|ccccc} & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ I_5 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$(K_{3,3}, ext4)$

Clearly, $(K_{3,3}, ext1) \cong M(K'_{3,3})$. It is shown in the Appendix that $(K_{3,3}, ext2) \cong M(E_5)$, $(K_{3,3}, ext3) \cong (P_9, ext7)^*$, and $(K_{3,3}, ext4) \cong R_{10}$. Theorem 2.3.1 implies that each of $M(K'_{3,3})$ and $(P_9, ext7)^*$ has an $M(K_5\backslash e)$-minor. Now if $M_1 \cong (K_{3,3}, ext1)$ or $(K_{3,3}, ext3)$, then it has an $M(K_5\backslash e)$-minor, and therefore so does $M$. So assume $M_1 \cong R_{10}$ or $M(E_5)$. Lemma 2.4.1 implies that $R_{10}$ and $M(E_5)$ are splitters for the class of matroids with no $M^*(K_5\backslash e)$- or $M(K_5\backslash e)$-minors. So the extensions and coextensions of $R_{10}$ and $M(E_5)$ have $M^*(K_5\backslash e)$- or $M(K_5\backslash e)$-minors, and therefore so does $M$.

The following corollary is a direct consequence of the previous proposition. The relations between the excluded-minor classes is shown in Figure 19.

**Corollary 2.4.3.** *The 3-connected binary matroids in* $EX[M(K_5\backslash e),\ M^*(K_5\backslash e)] - EX[M(K_5\backslash e),\ M^*(K_5\backslash e),\ M(K_{3,3}),\ M^*(K_{3,3})]$ *are precisely* $M(K_{3,3})$, $M^*(K_{3,3})$, $R_{10}$, *and* $M(E_5)$. □

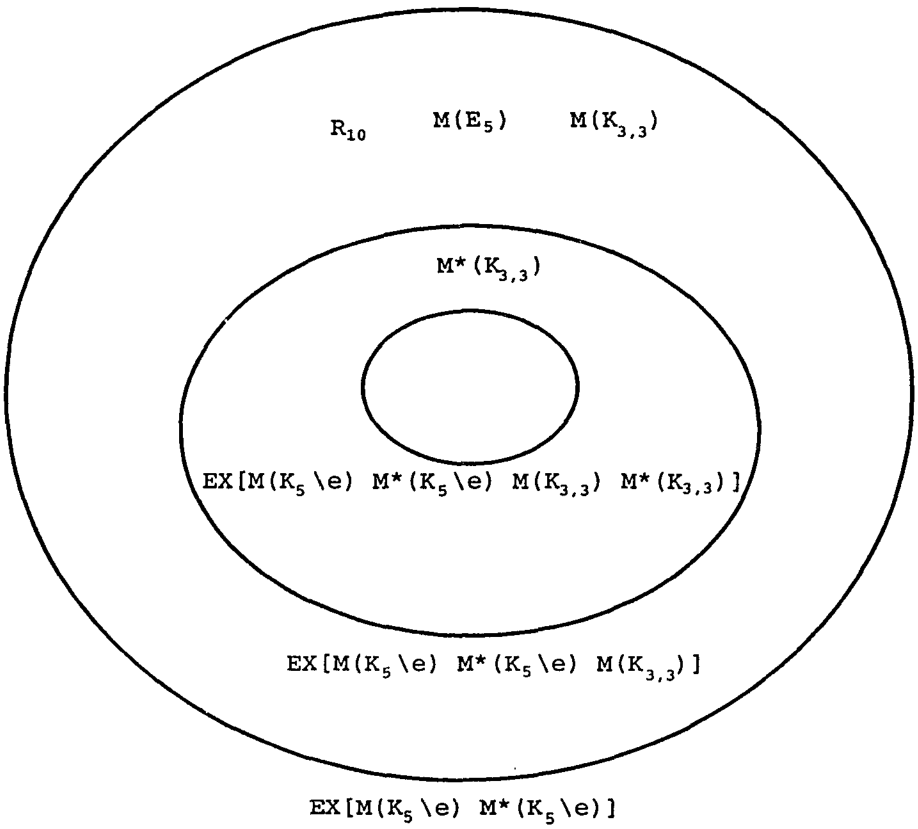



**Figure 19.** Some excluded minor classes

# CHAPTER III

# A GENERALIZATION OF D. W. HALL'S GRAPH RESULT

## 3.1. Motivation

In 1930, Kuratowski proved that a graph is planar if and only if it has no subgraph that is a subdivision of $K_5$ or $K_{3,3}$. Wagner (1937) proved that a graph is planar if and only if it has no minor isomorphic to $K_5$ or $K_{3,3}$. In 1943, Hall proved the following result which was stated earlier in a slightly different form as Theorem 1.1.4:

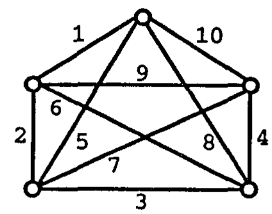
**Theorem 3.1.1** *Suppose $G$ is a simple, 3-connected graph with a $K_5$-minor. Then either $G$ has a $K_{3,3}$-minor, or $G \cong K_5$.*

Seymour (1980) noted the following result as an application of the Splitter Theorem.

**Theorem 3.1.2** *Suppose $M$ is a 3-connected regular matroid with an $M(K_5)$-minor. Then either $M$ has an $M(K_{3,3})$-minor, or $M \cong M(K_5)$.*

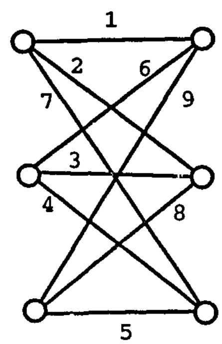
Theorem 3.2.1, the main result of this chapter, generalizes Hall's result to the class of binary matroids as follows: If a 3-connected binary matroid has a minor isomorphic to $M(K_5)$, then it must also have a minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$, the only exceptions being $M(K_5)$, a highly symmetric, 12-element matroid which we shall call $T_{12}$, and any single-element contraction of $T_{12}$. Observe that this result has the same flavor as Theorem 2.4.3.

$$\begin{array}{c} \begin{array}{cccc|cccccc} 1&2&3&4&5&6&7&8&9&10 \end{array}\\ \left(\begin{array}{c|cccccc} & 1&0&0&1&0&1\\ & 1&1&0&1&1&1\\ I_4 & 0&1&1&1&1&1\\ & 0&0&1&0&1&1 \end{array}\right) \end{array}$$




$M(K_5)$

$$\begin{array}{c} \begin{array}{ccccc|cccc} 1&2&3&4&5&6&7&8&9 \end{array}\\ \left(\begin{array}{c|cccc} & 1&0&0&1\\ & 1&1&0&1\\ I_5 & 1&1&1&1\\ & 0&1&1&1\\ & 0&0&1&1 \end{array}\right) \end{array}$$




$M(K_{3,3})$

$$\begin{array}{c} \begin{array}{cccccc|cccc} 5&6&7&8&9&10&1&2&3&4 \end{array}\\ \left(\begin{array}{c|cccc} & 1&1&0&0\\ & 0&1&1&0\\ & 0&0&1&1\\ I_6 & 1&1&1&0\\ & 0&1&1&1\\ & 1&1&1&1 \end{array}\right) \end{array}$$

$M^*(K_5)$

$$\begin{array}{c} \begin{array}{cccc|ccccc} 6&7&8&9&1&2&3&4&5 \end{array}\\ \left(\begin{array}{c|ccccc} & 1&1&1&0&0\\ & 0&1&1&1&0\\ I_4 & 0&0&1&1&1\\ & 1&1&1&1&1 \end{array}\right) \end{array}$$

$M^*(K_{3,3})$

$$\begin{array}{c} \begin{array}{cccccc|cccccc} 1&2&3&4&5&6&7&8&9&10&11&12 \end{array}\\ \left(\begin{array}{c|cccccc} & 1&1&0&0&0&1\\ & 1&0&0&0&1&1\\ & 0&0&0&1&1&1\\ I_6 & 0&0&1&1&1&0\\ & 0&1&1&1&0&0\\ & 1&1&1&0&0&0 \end{array}\right) \end{array}$$

$T_{12}$

**Figure 20.** The matroids $M(K_5)$, $M(K_{3,3})$, $M^*(K_5)$, $M^*(K_{3,3})$, and $T_{12}$.

We shall start with a description of the binary matroid $T_{12}$. It is defined to be the vector matroid of the binary matrix in Figure 20. Observe that columns 7 to 12 form a circulant matrix with three ones in each column. $T_{12}$ is self-dual, since the matrix formed by columns 7 to 12 is symmetric. $T_{12}$ is non-regular, since $T_{12}/4,6,8\backslash 1,9 \cong F_7$.

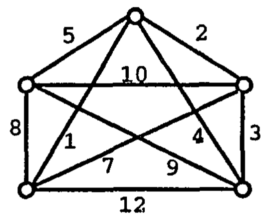
**Lemma 3.1.3** *$T_{12}$ has a transitive automorphism group.*

**Proof.** Cyclically permuting the rows of the matrix representing $T_{12}$ induces an automorphism on $T_{12}$ that swaps any pair of elements in each of the classes $\{1,2,3,4,5,6\}$ and $\{7,8,9,10,11,12\}$. The permutation of the set $\{1,2,\ldots,12\}$ that maps $(1,2,\ldots,12)$ to $(1,12,3,6,9,4,11,8,5,10,7,2)$ is an automorphism of $T_{12}$, that maps 2 to 12, and the lemma follows.

**Lemma 3.1.4** *$T_{12}$ has a minor isomorphic to $M(K_5)$ but has no minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$.*

**Proof.** Figure 21 shows that $T_{12}/6,11 \cong M(K_5)$. Since $T_{12}$ is self-dual it is sufficient to show that it has no $M(K_{3,3})$-minor. Suppose $T_{12}$ has an $M(K_{3,3})$-minor. Then, since $M(K_{3,3})$ has rank 5 and 9 elements, $T_{12}\backslash e,f$ has an $M(K_{3,3})$-minor for some $e$, $f$ in $E(T_{12})$. We show in Table 12c in the Appendix that $T_{12}$ has 15 circuits of size 4. Moreover, every element in $T_{12}$ occurs in exactly 5 circuits of size 4, and every pair of elements is in at most 2 circuits of size 4. Since, for any matroid $M$, the circuits of $M\backslash e$ are the circuits of $M$ that do not contain $e$, the matroid $T_{12}\backslash e,f$ has at most $15-10+2=7$ circuits of size 4. This is a contradiction since $M(K_{3,3})$ has 9 circuits of size 4. Hence $T_{12}$ has no $M(K_{3,3})$-minor. □

$$\begin{array}{cccccccccc} 1 & 3 & 4 & 5 & 7 & 8 & 9 & 10 & 2 & 12 \end{array}$$
$$\left(\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$




$T_{12}/6,11$ $\qquad\qquad$ $M(K_5)$

**Figure 21.** $T_{12}/6,11 \cong M(K_5)$

Finally, the binary matrix shown below is another representation for $T_{12}$.

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$
$$\left(\begin{array}{c|cccccc} & 1 & 0 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 0 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

Represented this way, it is clear that $T_{12}/6\backslash 12$ is isomorphic to the 5-wheel, $M(W_5)$. The permutation of the set $\{1,2,\ldots,12\}$ that maps $(1,2,\ldots,12)$ to $(7,2,6,3,9,5,11,12,1,8,4,10)$ is a isomorphism from the above representation of $T_{12}$ to the representation for $T_{12}$ given in Figure 20.

## 3.2 The main theorem

Recall that if a matroid $M$ has $k$ single-element extensions, we denote them by $(M, ext1)$, $(M, ext2)$, $\ldots$, $(M, extk)$.

**Theorem 3.2.1.** *Suppose $M$ is a 3-connected binary matroid with an $M(K_5)$-minor. Then either $M$ has an $M(K_{3,3})$- or $M^*(K_{3,3})$- minor, or $M$ is isomorphic to $M(K_5)$, $T_{12}$, or $T_{12}/e$.*

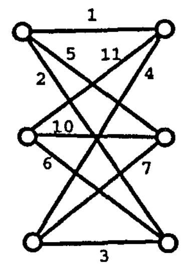
**Proof.** $M(K_5)$ has no minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$. By Lemma 3.1.3 there is up to isomorphism a unique single-element contraction of $T_{12}$. This contraction will be denoted by $T_{12}/e$. Lemma 3.1.4 implies that $T_{12}$, and hence $T_{12}/e$, has no minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$. Conversely, suppose $M$ is a 3-connected, binary matroid with an $M(K_5)$-minor. Theorem 1.2.2 implies that there is a chain $M_0$, $M_1$, $\ldots$, $M_n$ of 3-connected matroids such that $M_0 \cong M(K_5)$, $M_n = M$, and for all $i \in \{0, 1, \ldots, n-1\}$, $M_{i+1}$ is a single-element extension or coextension of $M_i$. If $M \cong M(K_5)$ then we have nothing to prove. Therefore, assume this does not occur. Then $M_0 \cong M(K_5)$ and $M_1$ is an extension or coextension of $M(K_5)$. Table 8b in the Appendix implies that $M(K_5)$ has precisely one binary, 3-connected, single-element extension, a matrix representation for which is shown below. Figure 22 shows that $(K_5, ext1)$ has an $M^*(K_{3,3})$-minor.

$$\begin{array}{c|ccccccc}
1\ 2\ 3\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}$$
$$\left(\begin{array}{c|ccccccc}
 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
I_4 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{array}\right)$$

$(K_5, ext1)$

$$\begin{array}{c} 5\ 6\ 7\ 10\ 11 \quad 1\ 2\ 3\ 4 \\ \left(\begin{array}{c|cccc} & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 0 \\ I_5 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 \end{array}\right) \end{array}$$

$[(K_5, ext1)\backslash 8, 9]^*$




$M(K_{3,3})$

**Figure 22.** $[(K_5, ext1)\backslash 8, 9]^* \cong M(K_{3,3})$

We may now suppose that $M_1$ is a coextension of $M(K_5)$, which means that $M_1$ is the dual of an extension of $M^*(K_5)$. Table 9b in the Appendix implies that $M^*(K_5)$ has precisely five non-isomorphic, binary, 3-connected, single-element extensions. Matrix representations for these matroids are shown below:

$$\begin{array}{c} 1\ldots 6 \quad 7\ 8\ 9\ 10\ 11 \\ \left(\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 0 \\ I_6 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array}$$

$(K_5^*, ext1)$

$$\begin{array}{c} 1\ldots 6 \quad 7\ 8\ 9\ 10\ 11 \\ \left(\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 0 \\ I_6 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array}$$

$(K_5^*, ext2)$

$$\begin{array}{c} 1\ldots 6 \quad 7\ 8\ 9\ 10\ 11 \\ \left(\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array}$$

$(K_5^*, ext3)$

$$\begin{array}{c} 1\ldots 6 \quad 7\ 8\ 9\ 10\ 11 \\ \left(\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right) \end{array}$$

$(K_5^*, ext4)$

$$\begin{array}{c|ccccc} 1\ldots 6 & 7 & 8 & 9 & 10 & 11 \\ & & & & & \end{array}$$

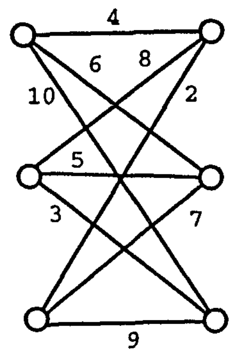
$$\left(\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 0 \\ I_6 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$(K_5^*, ext5)$

Using the matrix representation of $M(K_{3,3})$ in Figure 20, we see that each of $(K_5^*, ext1)/1,2$, $(K_5^*, ext2)/1,2$, and $(K_5^*, ext3)/1,2$ is isomorphic to $M^*(K_{3,3})$. Figure 23 shows that $(K_5^*, ext4)$ has an $M(K_{3,3})$-minor:

$$\begin{array}{c} \begin{array}{ccccc|cccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccc} & 1 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \end{array}\right) \end{array}$$

$(K_5^*, ext4)/11\backslash 1$




$M(K_{3,3})$

**Figure 23.** $(K_5^*, ext4)/11\backslash 1 \cong M(K_{3,3})$

Therefore $(K_5^*, ext5)^*$ is the only coextension of $M(K_5)$ with no minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$. Note that adjoining any one of the following columns to the matrix representing $M^*(K_5)$ gives $(K_5^*, ext5)$:

$$(101100)^T,\ (101010)^T,\ (011010)^T,\ (010110)^T,\ (000111)^T,$$
$$(111110)^T,\ (010101)^T,\ (100011)^T,\ (010011)^T,\ (101111)^T,$$
$$(110100)^T,\ (001101)^T. \qquad (3.2.2)$$

Without loss of generality, we will assume that $(K_5^*, ext5)$ is obtained by adjoining column $(001101)^T$ to the matrix representing $M^*(K_5)$. Observe that the following sequence of row operations applied to the matrix $(K_5^*, ext5)$ results in the matrix $T_{12}\backslash e$: First, pivot on element $[a_{6,11}]$, then on $[a_{1,7}]$, and finally on $[a_{3,6}]$. Switch row 1 with row 2, and row 5 with row 6. Therefore, the permutation of $\{1, 2, \ldots, 11\}$ that maps $(1, 2, \ldots, 11)$ to $(2, 7, 6, 4, 11, 5, 8, 9, 10, 3, 1)$ is an isomorphism from $(K_5^*, ext5)$ to $T_{12}\backslash e$. Since $T_{12}$ is self-dual, $(K_5^*, ext5)^* \cong T_{12}/e$. Therefore $M_1 \cong T_{12}/e$. Now $M_2$ is an extension or coextension of $T_{12}/e$. The next two lemmas determine which extensions and coextensions of $T_{12}/e$ have $M(K_{3,3})$- or $M^*(K_{3,3})$-minors.

**Lemma 3.2.3.** *Every binary, 3-connected extension of $T_{12}/e$ has a minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$.*

**Proof.** Table 10b in the Appendix implies that $T_{12}/e$ has four binary, 3-connected, single-element extensions. Matrix representations for these matroids are shown below:

$$\begin{array}{cc|ccccccc} & 1\ldots5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$
$$\left(\begin{array}{c|ccccccc} & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ I_5 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right)$$

$(T_{12}/e, ext1)$

$$\begin{array}{cc|ccccccc} & 1\ldots5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$
$$\left(\begin{array}{c|ccccccc} & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ I_5 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right)$$

$(T_{12}/e, ext2)$

$$\begin{array}{cc|ccccccc} & 1\ldots5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$
$$\left(\begin{array}{c|ccccccc} & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ I_5 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$$

$(T_{12}/e, ext3)$

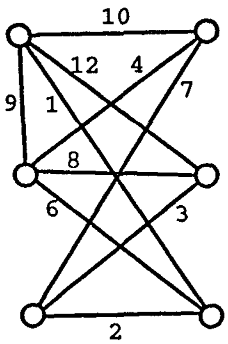
$$\begin{array}{cc|ccccccc} & 1\ldots5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$
$$\left(\begin{array}{c|ccccccc} & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ I_5 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$$

$(T_{12}/e, ext4)$

Clearly each of $(T_{12}/e, ext1)\backslash 6,7,9$, $(T_{12}/e, ext2)\backslash 6,7,9$, and $(T_{12}/e, ext3)\backslash 6,7,9$ is isomorphic to $M(K_{3,3})$. Figure 24 shows that $(T_{12}/e, ext4)$ has an $M^*(K_{3,3})$-minor. □

$$\begin{array}{c} \begin{array}{ccccccccccc} 6 & 7 & 9 & 10 & 12 & & 1 & 2 & 3 & 4 & 8 \end{array} \\ \left(\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 1 \end{array}\right) \end{array}$$

$[(T_{12}/e, ext4)\backslash 5, 11]^*$




$M(K'_{3,3})$

**Figure 24.** $[(T_{12}/e, ext4)\backslash 5, 11]^* \cong M(K'_{3,3})$

**Lemma 3.2.4.** $T_{12}$ *is the only binary, 3-connected, single-element extension of* $T_{12}\backslash e$ *with no minor isomorphic to* $M(K_{3,3})$ *or* $M^*(K_{3,3})$.

**Proof.** The columns listed in 3.2.2 are the only columns we need consider adding to the matrix representing $T_{12}\backslash e$, namely $(K_5^*, ext5)$. Table 11b in the appendix implies that adjoining any one of the first five columns listed in 3.2.2 gives $(T_{12}\backslash e, ext10)$; adjoining any one of the next five columns gives $(T_{12}\backslash e, ext11)$; and adjoining the last column gives $(T_{12}\backslash e, ext12)$. Matrix representations for these matroids are shown below:

$$\begin{array}{c} \begin{array}{ccccccc} 1\ldots 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \left(\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 0 & 0 \\ I_6 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right) \end{array} \qquad \begin{array}{c} \begin{array}{ccccccc} 1\ldots 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \left(\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 0 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right) \end{array}$$

$(T_{12}\backslash e, ext10)$ $\qquad$ $(T_{12}\backslash e, ext11)$

$$
\begin{array}{c}
\begin{array}{cc|cccccc} \quad 1\ldots6 \quad & & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\left(\begin{array}{c|cccccc}
 & 1 & 1 & 0 & 0 & 0 & 1 \\
 & 0 & 1 & 1 & 0 & 0 & 1 \\
I_6 & 0 & 0 & 1 & 1 & 1 & 0 \\
 & 1 & 1 & 1 & 0 & 1 & 1 \\
 & 0 & 1 & 1 & 1 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right)
\end{array},
$$

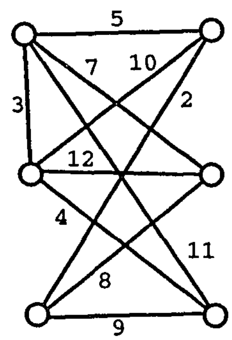
$(T_{12}\backslash e, ext12)$

Figure 25 shows that each of $(T_{12}\backslash e, ext10)$ and $(T_{12}\backslash e, ext11)$ has an $M(K_{3,3})$-minor. Observe that the same sequence of operations that were applied to $(K_5^*, ext5)$ to get $T_{12}\backslash e$, when applied to $(T_{12}\backslash e, ext12)$, gives $T_{12}$. Lemma 3.1.4 implies that $T_{12}$ has no minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$.

□

$$
\begin{array}{c}
\begin{array}{ccccc|ccccc} 7 & 2 & 3 & 4 & 5 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\left(\begin{array}{c|ccccc}
 & 1 & 0 & 0 & 0 & 1 \\
 & 1 & 1 & 0 & 0 & 0 \\
I_5 & 0 & 1 & 1 & 1 & 1 \\
 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 1 & 1 & 0 & 0
\end{array}\right)
\end{array}
$$

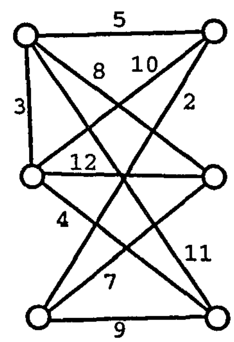
$(T_{12}\backslash e, ext10)/6\backslash 1$




$M(K'_{3,3})$

$$
\begin{array}{c}
\begin{array}{ccccc|ccccc} 7 & 2 & 3 & 4 & 5 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\left(\begin{array}{c|ccccc}
 & 1 & 0 & 0 & 0 & 1 \\
 & 1 & 1 & 0 & 0 & 1 \\
I_5 & 0 & 1 & 1 & 1 & 1 \\
 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 1 & 1 & 0 & 1
\end{array}\right)
\end{array}
$$

$(T_{12}\backslash e, ext11)/6\backslash 1$




$M(K'_{3,3})$

**Figure 25.** $(T_{12}\backslash e, ext10)/6\backslash 1 \cong M(K'_{3,3})$ and $(T_{12}\backslash e, ext11)/6\backslash 1 \cong M(K'_{3,3})$

Since $T_{12}$ is self-dual, the previous lemma implies that $T_{12}$ is the only binary, 3-connected coextension of $T_{12}/e$ with no $M(K_{3,3})$- or $M^*(K_{3,3})$-minor. The following corollary, which is a direct consequence of the previous lemma and the fact that $T_{12}$ is self-dual, completes the proof of the theorem.

**Corollary 3.2.5.** *$T_{12}$ is a splitter for the class of binary matroids with no minor isomorphic to $M(K_{3,3})$ or $M^*(K_{3,3})$.* $\square$

Observe that Theorems 3.1.1 and 3.1.2 may be obtained as corollaries of the above theorem. This is because the only binary, 3-connected, single-element extension of $M(K_5)$, namely $(K_5, ext1)$, is non-regular since $(K_5, ext1)/4\backslash 7, 10, 11 \cong F_7$. The matroid $(K_5^*, ext1)^*$ is the only regular coextension of $M(K_5)$ and it is isomorphic to the graph $K_{3,3}$ with two additional edges, one in each vertex class. Clearly it has a minor isomorphic to $M(K_{3,3})$. The remaining four coextensions of $M(K_5)$ are non-regular.

Determining the structure of excluded-minor classes is an interesting and well worked problem. For example, $EX[M^*(K_5),\ M^*(K_{3,3}),\ F_7,\ F_7^*]$ is precisely the class of graphic matroids, and $EX[M(K_5),\ M(K_{3,3}),\ F_7,\ F_7^*]$ is the class of cographic matroids. Therefore it is natural to determine the class of binary matroids with no minors isomorphic to the Kuratowski graphs, namely, $EX[M(K_5),\ M(K_{3,3})]$. This problem is unsolved. Kung (1987) has determined bounds on the number of elements in $EX[M(K_{3,3})]$ and $EX[M(K_5)]$. The exact members of these classes are still unknown. In fact, the members of the following classes are unknown: $EX[M(K_5),\ M^*(K_5)]$, $EX[M(K_{3,3}),\ M^*(K_{3,3})]$, and $EX[M(K_5),\ M^*(K_5),\ M(K_{3,3}),\ M^*(K_{3,3})]$. The following corollary, which is a direct consequence of the previous theorem, notes that the last two classes are almost the same.

**Corollary 3.2.6.** *The 3-connected binary matroids in* $EX[M(K_{3,3}),\ M^*(K_{3,3})]\ -\ EX[M(K_5),\ M^*(K_5),\ M(K_{3,3}),\ M^*(K_{3,3})]$ *are precisely* $M(K_5)$, $M^*(K_5)$, $T_{12}$, $T_{12}/e$, *and* $T_{12}\backslash e$. □

**Corollary 3.2.7.** *The 3-connected binary matroids in* $EX[M(K_{3,3}),\ M^*(K_{3,3})]\ -\ EX[M(K_5),\ M(K_{3,3}),\ M^*(K_{3,3})]$ *are precisely* $M(K_5)$, $T_{12}$, *and* $T_{12}/e$. □

The relations between the various excluded-minor classes considered above are depicted in Figure 26.

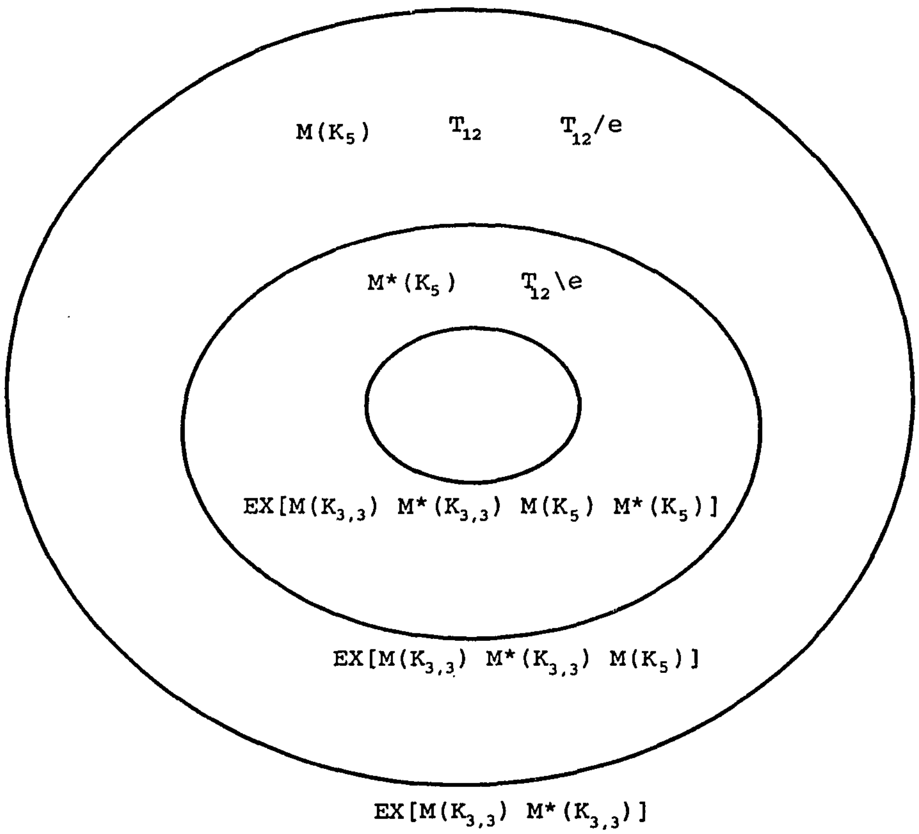



**Figure 26.** Some excluded minor classes

# CHAPTER IV

# INTERSECTIONS OF CIRCUITS AND COCIRCUITS IN BINARY MATROIDS

## 4.1. Motivation

A set with $k$ elements which is the intersection of a circuit and a cocircuit in a matroid will be called a *special $k$-set.* The following results are well-known: A circuit and a cocircuit in a matroid cannot have exactly one common element. In a connected matroid, every pair of elements is a special 2-set. A matroid is binary if and only if every set which is the intersection of a circuit and a cocircuit has even cardinality. We will start by surveying existing results on special sets. The first two results may be found in Seymour (1976, 1981). The remaining results may be found in Oxley (1984). Note that Seymour calls a special 3-set a *triad*, and Oxley calls a special 4-set a *quad.*

**Theorem 4.1.1.** *A matroid has no special* 3*-set if and only if it is binary.* □

**Theorem 4.1.2.** *If a connected matroid $M$ has a special* 3*-set, then every pair of elements of $M$ is in a special* 3*-set.* □

**Theorem 4.1.3.** *A connected matroid has no special* 4*-set if and only if it is a series-parallel network.* □

**Theorem 4.1.4.** *If a* 3*-connected matroid $M$ has a special* 4*-set, then every pair of elements is in a special* 4*-set.* □

**Proposition 4.1.5.** *A binary matroid has a special* 4*-set if and only if it has an $M(K_4)$-minor.* □

**Theorem 4.1.6.** *Let $M$ be a matroid with a special $k$-set, for some $k \geq 4$. Then $M$ has a special 4-set.* $\square$

**Proposition 4.1.7.** *Let $N$ be a minor of $M$, and suppose that $X$ is a special $k$-set in $N$. Then $X$ is a special $k$-set in $M$.* $\square$

**Proposition 4.1.8.** *Let $M$ be a matroid containing a special $k$-set $X$. Then $M$ has a minor $N$ in which $X$ is both a circuit and a cocircuit and $r(N) = r^*(N) = k - 1$.* $\square$

We will make frequent use of the following result on binary matroids (Oxley, 1992, 9.1.4), which describes the behaviour of circuits in binary matroids. The *symmetric difference*, $A \triangle B$, of two sets $A$ and $B$ equals $(A - B) \cup (B - A)$.

**Proposition 4.1.9.** *If $C_1$ and $C_2$ are circuits in a binary matroid, then $C_1 \triangle C_2$ is a disjoint union of circuits.* $\square$

Oxley (1992, 14.8.3) has made the following conjecture about special sets:

**Conjecture 4.1.10.** *If a matroid has a special $k$-set, for some $k \geq 4$, then it has a special $(k-2)$-set.* $\square$

In Section 4.2 we prove this conjecture for graphic matroids. In Section 4.3, we prove that if a binary matroid has a special $k$-set, for some $k \geq 6$, then it must have a special 6-set. This result extends Theorem 4.1.6 in the binary case. While proving this result, we show that if a binary matroid has a special 8-set, then it must have a special 6-set, thereby lending more credibility to Conjecture 4.1.10. Finally, in Section 4.4, we determine all the regular matroids with no special 6-set.

## 4.2. Intersections of circuits and cocircuits in graphic matroids

**Lemma 4.2.1.** *Let $M(G)$ be a graphic matroid with a special $k$-set $X$, for some $k \geq 4$. Then $M(G)$ has a connected minor $M(H)$ such that:*

(i) *$X$ is both a circuit and a cocircuit in $M(H)$, $r(M(H)) = r^*(M(H)) = k-1$, and $|E(M(H))| = 2(k-1)$.*

(ii) *$H - X$ has two connected components $T_1$ and $T_2$, each of which is a tree with $k/2$ vertices and $(k-2)/2$ edges, such that, every edge in $X$ has one end-vertex in $T_1$ and the other in $T_2$ (see Figure 27).*

**Proof.** Part (i) follows from Proposition 4.1.8 and the fact that $|E(M(H))| = r(M(H)) + r^*(M(H))$. Since $X$ is a spanning circuit of $M(H)$, this matroid is connected. We may assume that $H$ has no isolated vertices. Next, since $r(M(H)) = k-1$ and $|X| = k$, the set $X$ is a spanning set and $H - X$ has no circuits. Therefore $H - X$ is a forest. Since $X$ is a cocircuit in $M(H)$, it is a minimal cutset in $H$. Therefore $H - X$ has exactly two connected components, say $T_1$ and $T_2$, and each edge in $X$ has one end-vertex in $T_1$ and the other in $T_2$. Observe that $T_1$ and $T_2$ are trees. Finally, since $X$ is a spanning circuit with $k$ edges, each vertex of $H$ is incident with exactly two edges of $X$. Therefore, each of $T_1$ and $T_2$ has $k/2$ vertices and $(k-2)/2$ edges. $\square$

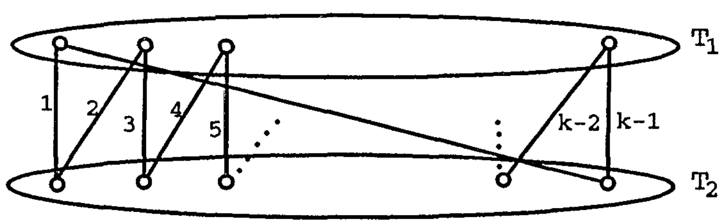



**Figure 27.** The minor $H$ in a graph with a special $k$-set

**Theorem 4.2.2.** *Let $M(G)$ be a graphic matroid with a special $k$-set $X$, for some $k \geq 4$. Then $M(G)$ has at least four special $(k-2)$-sets contained in $X$.*

**Proof.** The result holds for $k = 4$, since every pair of elements in a connected graph is a special 2-set. Therefore assume $k \geq 6$. Lemma 4.2.1 implies that $M(G)$ has a minor $M(H)$ with rank and corank equal to $k-1$, such that $H - X$ has two connected components $T_1$ and $T_2$, each of which is a tree. Each of $T_1$ and $T_2$ has at least two leaves, that is, vertices of degree 1. We shall show that each leaf in $T_1$ and $T_2$ gives rise to a special $(k-2)$-set contained in $X$, and that different leaves yield different special $(k-2)$-sets.

Let $v$ be a leaf in $T_1$ or $T_2$, say $T_1$. From Figure 27, we see that $v$ is incident with exactly two edges of $X$. Therefore the degree of $v$ in $H$ is 3. Let $C_v^*$ be the set of edges incident on $v$. Since $M(H)$ is connected, $C_v^*$ is a cocircuit of size three. Let $e$ be the edge of $T_1$ incident on $v$. Proposition 4.1.9 implies that the set $X \triangle C_v^*$ is a disjoint union of cocircuits. Since $X \triangle C_v^* = (X - C_v^*) \cup e$, it is a cocircuit. Observe that $X - C_v^*$ is the intersection of the cocircuit $(X - C_v^*) \cup e$ and the circuit $X$, and $|X - C_v^*| = k-2$. Therefore $X - C_v^*$ is a special $(k-2)$-set in $M(H)$, and hence in $M(G)$.

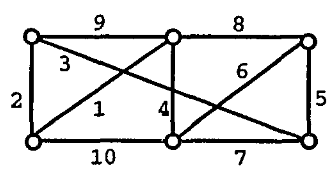
It remains to show that different leaves yield different special $(k-2)$-sets. Let $v_1$ and $v_2$ be any two leaves of $T_1$. Let $C_1^*$ and $C_2^*$ be the sets of edges incident on $v_1$ and $v_2$, respectively. Then $C_1^* \cap C_2^*$ is non-empty only if $v_1$ and $v_2$ are adjacent in $T_1$. But, since $k \geq 6$, $T_1$ has at least three vertices, so the leaves $v_1$ and $v_2$ are nonadjacent. Therefore $C_1^* \cap C_2^*$ is empty and so $X - C_1^* \neq X - C_2^*$. Finally, let $v_1$ and $v_2$ be leaves of $T_1$ and $T_2$, respectively. Then, for $i = 1, 2$, both edges in $C_i^* \cap X$ have $v_i$ as an end-vertex. Hence $|C_1^* \cap C_2^* \cap X| \leq 1$, so $C_1^* \cap X$ and $C_2^* \cap X$ are distinct, and again, $X - C_1^* \neq X - C_2^*$. □

### 4.3. The binary matroids with special 6-sets

The main result of this section, Theorem 4.3.3, states that if a binary matroid $M$ has a special $k$-set, for some $k \geq 6$, then $M$ has a special 6-set. In addition, an included-minor characterization for when a binary matroid has a special 6-set is proved. In Figure 28, binary matrices representing each of $M(G_{10})$, $M(K_4) \oplus_2 M(K_4)$, $M(A_5)\backslash a_6$, and $M(Z_5)\backslash b_5$, together with graphs representing $M(G)$ and $M(K_4) \oplus_2 M(K_4)$ are given.

$$\begin{array}{c}
\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&10\end{array}\\
\left(\begin{array}{c|ccccc} & 1&1&1&1&1\\ & 1&1&1&1&0\\ I_5 & 1&1&1&0&0\\ & 1&1&0&0&1\\ & 1&0&1&0&0\end{array}\right)
\end{array}$$

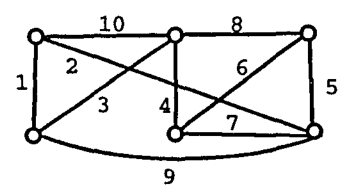
$M(G_{10})$




$(K_5\backslash e)^* + edge$

$$\begin{array}{c}
\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&10\end{array}\\
\left(\begin{array}{c|ccccc} & 1&1&1&1&1\\ & 1&1&1&1&0\\ I_5 & 1&1&1&0&1\\ & 1&1&0&0&0\\ & 1&0&1&0&0\end{array}\right)
\end{array}$$

$M(K_4) \oplus_2 M(K_4)$




$K_4 \oplus_2 K_4$

$$\begin{array}{c}
\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&10\end{array}\\
\left(\begin{array}{c|ccccc} & 1&1&1&1&1\\ & 1&1&1&1&0\\ I_5 & 1&1&1&0&1\\ & 1&1&0&1&0\\ & 1&0&1&1&0\end{array}\right)
\end{array}$$

$M(A_5)\backslash a_6$

$$\begin{array}{c}
\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&10\end{array}\\
\left(\begin{array}{c|ccccc} & 1&0&1&1&1\\ & 1&1&0&1&1\\ I_5 & 1&1&1&0&1\\ & 1&1&1&1&0\\ & 1&1&1&1&1\end{array}\right)
\end{array}$$

$M(Z_5)\backslash b_5$

**Figure 28.** $M(G_{10})$, $M(K_4) \oplus_2 M(K_4)$, $M(A_5)\backslash a_6$, and $M(Z_5)\backslash b_5$

$M(G_{10})$ is, in fact, the prism graph $(K_5\backslash e)^*$ with an edge added. Up to isomorphism there is exactly one such simple graph. $M(K_4) \oplus_2 M(K_4)$ is the 2-sum of $M(K_4)$ with itself. $M(A_5)\backslash a_6$ is a member of the infinite family described in Chapter 2. It is also isomorphic to $(P_9^*, ext1)$. The permutation of $\{1, 2, \ldots, 10\}$ that maps $(1, 2 \ldots, 10)$ to $(8, 4, 1, 9, 2, 5, 7, 3, 10, 6)$ is an isomorphism from the matrix representing $M(A_5)\backslash a_6$ in Figure 28 to $(P_9^*, ext1)$. The matroid $M(Z_5)\backslash b_5$ is a member of the infinite family in Theorem 1.1.7. The following proposition is similar to Proposition 4.1.5.

**Proposition 4.3.1.** *A binary matroid $M$ has a special 6-set if and only if $M$ has a minor isomorphic to $M(G_{10})$, $M(K_4) \oplus_2 M(K_4)$, $M(A_5)\backslash a_6$, or $M(Z_5)\backslash b_5$.*

**Proof.** Observe that each of the matroids $M(G_{10})$, $M(K_4) \oplus_2 M(K_4)$, $M(A_5)\backslash a_6$, and $M(Z_5)\backslash b_5$ has a special 6-set, namely, $\{1, 2, 3, 4, 5, 6\}$. Therefore, if $M$ has a minor isomorphic to one of the above matroids, then Proposition 4.1.7 implies that $M$ must have a special 6-set. Conversely, let $M$ be a binary matroid with a special 6-set $X$. Then Proposition 4.1.8 implies that $M$ has a minor $N$ in which $X$ is a circuit and a cocircuit, and such that $r(N) = r^*(N) = 5$. Therefore, $N$ is a 10-element, rank-5, simple, cosimple matroid. A partial binary matrix representation $A$, for $N$, of the form $[I_5|D]$, is shown below. Without loss of generality, we may assume that $X = \{1, 2, 3, 4, 5, 6\}$, so the first five columns form a basis. Then 6 must be a column of ones. Since $X$ is also cospanning, the set of column vectors $\{\bar{7}, \bar{8}, \bar{9}, \bar{10}\}$ is independent. Moreover, since each $\bar{i}$ in this set is the incidence vector of the set $C(i, \{1, 2, 3, 4, 5\}) - i$, and $C(i, \{1, 2, 3, 4, 5\})$ must meet the cocircuit $X$ in a set of even cardinality, $\bar{i}$ must have exactly two ones or exactly four ones.

$$A = \begin{array}{c} \begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{ccccc|ccccc} & & & & & 1 & & & & \\ & & & & & 1 & & & & \\ & & I_5 & & & 1 & & & & \\ & & & & & 1 & & & & \\ & & & & & 1 & & & & \end{array}\right) \end{array}$$

We shall first assume that each of the column vectors $\bar{7}$, $\bar{8}$, $\bar{9}$, and $\bar{10}$ has four ones. Then $N \cong M(Z_5)\backslash b_5$. Next, suppose that exactly three column vectors in $\{\bar{7}, \bar{8}, \bar{9}, \bar{10}\}$ have four ones, say $\bar{7} = (11110)^T$, $\bar{8} = (11101)^T$, and $\bar{9} = (11011)^T$. If the first two entries of $\bar{10}$ are both one or both zero, then the first two rows of $A$ would be identical, so $\bar{10}$ may be $(10100)^T$, $(10010)^T$, or $(10001)^T$. In each case, $N \cong M(A_5)\backslash a_6$.

Suppose that two column vectors in $\{\bar{7}, \bar{8}, \bar{9}, \bar{10}\}$ have four ones, say $\bar{7} = (11110)^T$ and $\bar{8} = (11101)^T$, while $\bar{9}$ and $\bar{10}$ have two ones each. Based on the symmetry of the existing columns, and the fact that no two rows in $D$ are identical, there are four choices for the pair $\{\bar{9}, \bar{10}\}$, namely $\{(11000)^T, (10100)^T\}$, $\{(11000)^T, (10010)^T\}$, $\{(10010)^T, (01010)^T\}$, or $\{(10010)^T, (01001)^T\}$. In the first case, $N \cong M(K_4) \oplus_2 M(K_4)$. It can be checked that in the second and fourth cases, $N \cong M(G_{10})$, and in the third case, $N \cong M(A_5)\backslash a_6$.

Suppose that one column vector in $\{\bar{7}, \bar{8}, \bar{9}, \bar{10}\}$ has four ones, say $\bar{7} = (11110)^T$. Then since $M^*$ has no parallel elements, each row in $D$ must have at least two ones. In particular, row 5 must have at least two ones. Pivoting on element $[a_{5,6}]$ gives a matrix in which at least two of the last four columns have four ones, and this case is already done. Finally, suppose that none of the column vectors in $\{\bar{7}, \bar{8}, \bar{9}, \bar{10}\}$ has four ones. Then once again using an argument similar to the previous one we can get a matrix in which at least one of the last four columns has four ones and this case is already done. Hence proved. $\square$

**Corollary 4.3.2.** *A regular matroid has a special* 6*-set if and only if it has a minor isomorphic to* $M(G_{10})$ *or* $M(K_4) \oplus_2 M(K_4)$.

**Proof.** The proof follows from the previous proposition and the fact that $M(A_5)\backslash a_6$ and $M(Z_5)\backslash b_5$ are not regular. □

The next theorem is the main result of this section and is similar to Theorem 4.1.6.

**Theorem 4.3.3.** *Let $M$ be a binary matroid with a special $k$-set $X$, for some $k \geq 6$. Then $M$ has a special* 6*-set contained in $X$.*

Before proving this result we shall state two lemmas, the first of which may be found in Oxley (1984, (2.6)).

**Lemma 4.3.4.** *Let $M$ have a special $k$-set $X$. Then, for some $t \in \{\lceil k/2 \rceil, \lceil k/2 \rceil + 1, \ldots, k-1\}$, $M$ has a special $t$-set in $X$.* □

**Lemma 4.3.5.** *Let $M$ be a binary matroid with a special* 8*-set $X$. Then $M$ has a special* 6*-set contained in $X$.*

**Proof.** Proposition 4.1.8 implies that $M$ has a minor $N$ in which $X$ is a special 8-set and $r(N) = r^*(N) = 7$. Therefore $N$ is a 10-element, rank-7 binary matroid. Consider a standard representation for $N$ of the form $[I_7|D]$. We may assume that $X = \{1, 2, \ldots, 8\}$, so 8 is a column of ones. Since $X$ is cospanning, the set of columns $Y = \{9, 10, \ldots, 14\}$ is independent. Moreover, as in the proof of Proposition 4.3.1, each column in $Y$ may have 2, 4, or 6 ones. We shall show that, in all cases, $N$ has a special 6-set contained in $X$. Let $i \in Y$. Suppose $i$ has 6 ones, say, $\bar{i} = (1111110)^T$. Then $\{1, 2, 3, 4, 5, 6, i\}$ is a circuit whose intersection with the cocircuit $X$ has 6 elements. Next, suppose $i$ has 2 ones, say, $\bar{i} = (1100000)^T$. Then $\{1, 2, i\}$ is a 3-circuit in $N$. Since $N$ is binary, Proposition 4.1.9 implies that $X \triangle C$ is a disjoint union of circuits. However, since $X \triangle C = (X - C) \cup i$, it is a circuit. The intersection of $X \triangle C$ with $X$ has 6 elements. We may now assume that all the columns in $Y$ have exactly four ones. Without loss of generality, assume that $\bar{i} = (1111000)^T$. A pair of columns may have 1, 2, or 3 ones in common. If all the columns in $Y - i$ meet $i$ in exactly 2 ones, then in the dual $\{1, 2, 3, 4\}$ would be a circuit. This is a contradiction since $\{1, 2, 3, 4\}$ is contained in $X$, which is a cocircuit. So there is a column $j$ that has 1 or 3 ones in common with $i$. First suppose that $\bar{j} = (1000111)^T$. Then $\{1, 8, i, j\}$ is a circuit. Since $N$ is binary, $X \triangle \{1, 8, i, j\}$, which is $\{2, 3, 4, 5, 6, 7, i, j\}$, is a disjoint union of circuits. Suppose there is a circuit $C$ properly contained in $\{2, 3, 4, 5, 6, 7, i, j\}$. Then $i$ or $j \in C$, say $i \in C$. However, the only circuits in $X \cup i$ containing $i$ are $\{1, 2, 3, 4, i\}$ and $\{5, 6, 7, 8, i\}$. So $i \notin C$ and similarly $j \notin C$. Therefore $C$ is properly contained in $X$, which is a contradiction. Hence $\{2, 3, 4, 5, 6, 7, i, j\}$ is a circuit and its intersection with $X$ has 6 elements. Finally, if $j$ has 3 ones in common with $i$, we may assume that $\bar{j} = (1110100)^T$. Then $\{4, 5, i, j\}$ is a circuit and by an argument similar to the previous one, we will find a special 6-set in $X$. □

**Proof of Theorem 4.3.3.** The proof is by induction on $k$. If $k = 6$, the result is immediate. If $k = 8$, then Lemma 4.3.5 implies that a matroid with a special 8-set $X$, must have a special 6-set contained in $X$. Let $k \geq 10$. Assume that the result is true if $M$ has a special $t$-set, for some $t < k$, and let $M$ have a special $k$-set $X$. Lemma 4.3.4 implies that $M$ has a special $t$-set $Y$, for some $t \in \{\lceil k/2 \rceil, \lceil k/2 \rceil + 1, \ldots, k - 1\}$. Since $M$ is binary, $t$ must be even, and since

$k \geq 10$, we have $t \geq 6$. By the induction hypothesis, $M$ has a special 6-set contained in $Y$, and therefore in $X$. $\square$

Theorem 4.1.4 implies that if $M$ is a 3-connected matroid containing a special 4-set, then every pair of elements is in a special 4-set. To see that a similar result does not hold if we replace "special 4-set" by "special 6-set", consider the matroid $M(G_{10})$ shown in Figure 28. It is 3-connected but has only one special 6-set, namely $\{1, 2, 3, 4, 5, 6\}$.

## 4.4. The regular matroids without special 6-sets

In this section, the regular matroids without special 6-sets will be determined. Let $\mathcal{M}= \{M(W_r)\ (r \geq 3),\ M(K_5),\ M(K_5\backslash e),\ M(K_{3,p}),\ M(K'_{3,p}),$ $M(K''_{3,p}),\ M(K'''_{3,p})\ (p \geq 3),\ R_{10}\}$. Let $\mathcal{M}^*$ be the set containing the duals of the above matroids. The following result is similar to Theorem 4.1.3.

**Theorem 4.4.1.** *$M$ is a connected regular matroid with no special 6-set if and only if $M$ is a series-parallel extension of a matroid in $\mathcal{M}$ or $\mathcal{M}^*$, or $M$ is a series-parallel network.*

**Proof.** Suppose $M$ is a series-parallel extension of a matroid in $\mathcal{M}$. In order to show that $M$ has no special 6-set, it is sufficient, by Corollary 4.3.2, to show that $M$ has no minor isomorphic to $M(G_{10})$ or $M(K_4) \oplus_2 M(K_4)$. $M$ has no minor isomorphic to $M(G_{10})$, because otherwise $M$ would have an $M^*(K_5\backslash e)$-minor, which contradicts Theorem 2.1.3. Since $M$ is a series-parallel extension of a 3-connected matroid in $\mathcal{M}$, by definition, $M$ has no minor isomorphic to $M(K_4) \oplus_2 M(K_4)$. Since the class of regular matroids without special 6-sets is closed under duality, series-parallel extensions of the matroids in $\mathcal{M}^*$ have no special 6-sets. Finally, Theorem 4.1.3 implies that a series-parallel network has no special 6-set.

Conversely, suppose $M$ is a regular matroid with no special 6-set. We will first show that $M$ is either a series-parallel network or a series-parallel extension of a 3-connected matroid. Suppose $M$ is not a series-parallel network. If $M$ itself is 3-connected, then we have nothing to show. Otherwise Theorem 1.2.1 implies that $M \cong M_1 \oplus_2 M_2$ where $M_1$ and $M_2$ are isomorphic to proper minors of $M$, and $E(M_1) \cap E(M_2) = \{p\}$. Suppose both $M_1$ and $M_2$ have 3-connected minors with at least four elements. Then by a result of Seymour (1985) each of $M_1$ and $M_2$ has an $M(K_4)$-minor containing $p$. Therefore, $M$ has a minor isomorphic to $M(K_4) \oplus_2 M(K_4)$. This is a contradiction since $M(K_4) \oplus_2 M(K_4)$ has a special 6-set. Therefore one of $M_1$ or $M_2$ is a series-parallel network, say $M_2$, and $M$ is a series-parallel extension of $M_1$. If $M_1$ is not 3-connected, we can repeat the above argument until we find a 3-connected minor $N$ of $M$ such that $M$ is a series-parallel extension of $N$.

Finally, it remains to show that $N$ is in $\mathcal{M}$ or $\mathcal{M}^*$. Since $N$ is regular, Theorem 1.2.4 implies that $N$ is graphic, or cographic, or has a minor isomorphic to $R_{10}$ or $R_{12}$. $R_{12}$ has $M(G_{10})$ as a minor and the latter has a special 6-set. Therefore $N$ cannot have an $R_{12}$-minor. Since $R_{10}$ is a splitter for the class of regular matroids, if $N$ has an $R_{10}$-minor, then $N \cong R_{10}$ and so $N$ is in $\mathcal{M}$. Therefore we may assume that $N$ is graphic or cographic. We will first assume that $N$ is a 3-connected graphic matroid without special 6-sets. Suppose, if possible, $N$ has an $M^*(K_5 \backslash e)$-minor. Theorem 1.2.2 implies that $N$ has, as a minor, a single-element extension or coextension of $M^*(K_5 \backslash e)$. However, $M^*(K_5 \backslash e)$ has no graphic single-element coextension, and $M(G_{10})$ is the only graphic single-element extension of $M^*(K_5 \backslash e)$. So $N$ must have an

$M(G_{10})$-minor. This is a contradiction. Therefore $N$ has no $M^*(K_5\backslash e)$-minor. Theorem 2.1.1 implies that $N$ is a graphic matroid in $\mathcal{M}$. By duality, if $N$ is a 3-connected cographic matroid without special 6-sets, then $M$ is a cographic matroid in $\mathcal{M}^*$. $\square$

Theorems 4.1.3 and 4.1.6 imply that series-parallel networks are the only matroids whose circuit and cocircuit intersections are of size at most two. Similarly, Theorems 4.4.1 and 4.3.3 imply that series-parallel networks and series-parallel extensions of the matroids in $\mathcal{M}$ and $\mathcal{M}^*$ are the only regular matroids whose circuit and cocircuit intersections are of size at most four.

# CHAPTER V

# AN EXCLUDED-MINOR CLASS OF MATROIDS

## 5.1. Motivation

In this chapter we present a "structure-driven" excluded-minor result. We will start with a description of the ideas and questions that motivated this result. In graph theory, there are two natural ways to define a graph $G$ that is minimal with a certain property: (i) $G$ has the property but $G \backslash e$ does not, for all edges e of $G$; and (ii) $G$ has the property but $G - v$ does not, for all vertices $v$ of $G$. The analogue of (i) for matroids is clear, but for (ii) it is not. This is because arbitrary matroids do not have vertices. However, in a graph, the edges incident with a vertex form a minimal cutset, that is, a cocircuit. Therefore, we may think of cocircuits as "vertices" in a matroid. If $C^*$ is a cocircuit of a matroid $M$ then its complement, $E(M) - C^*$, is a hyperplane. Thus deleting a vertex in $M$ is equivalent to restricting $M$ to the corresponding hyperplane.

For a class of matroids $\mathcal{M}$ closed under isomorphisms, direct sums, and the taking of minors, let $\mathcal{M}_1$ be the class of matroids with the property that a matroid is in $\mathcal{M}_1$ if its restriction to every hyperplane is in $\mathcal{M}$. In Section 5.2 we give necessary and sufficient conditions for a matroid to be in $\mathcal{M}_1$. In Section 5.3, we study the case when $\mathcal{M}$ is binary and obtain several characterizations for $\mathcal{M}_1$. Finally, in Section 5.4 we present an excluded-minor characterization for a more general class of matroids.

We will end this section by listing several characterizations for binary matroids, proofs for which may be found in Oxley (1992, 9.1). A pair of distinct circuits $C_1$ and $C_2$ form a *modular pair* if $r(C_1)+r(C_2) = r(C_1 \cup C_2)+r(C_1 \cap C_2)$

(White, 1987, p. 37). It follows that $r(C_1 \cup C_2) = |C_1| - 1 + |C_2| - 1 - |C_1 \cap C_2| = |C_1 \cup C_2| - 2$.

**Theorem 5.1.1.** *The following statements are equivalent for a matroid $M$.*

(i) *$M$ is binary.*

(ii) *$M$ has no $U_{2,4}$-minor.*

(iii) *If $C$ is a circuit and $C^*$ is a cocircuit, then $|C \cap C^*|$ is even.*

(iv) *If $C$ is a circuit and $C^*$ is a cocircuit, then $|C \cap C^*| \neq 3$.*

(v) *If $C_1$ and $C_2$ are distinct circuits, then $C_1 \triangle C_2$ is a disjoint union of circuits.*

(vi) *If $C_1$ and $C_2$ are distinct circuits, then $C_1 \triangle C_2$ contains a circuit.*

(vii) *The symmetric difference of any set of circuits is a disjoint union of circuits.*

(viii) *The symmetric difference of any set of circuits is either empty or contains a circuit.*

(ix) *If $C_1$ and $C_2$ are distinct circuits, and $e$ and $f$ are elements of $C_1 \cap C_2$, then $(C_1 \cup C_2) - \{e, f\}$ contains a circuit.*

(x) *If $C_1$, $C_2$, and $C_3$ are distinct circuits, such that $C_1 \cap C_2 \cap C_3 \neq \emptyset$, then there are distinct elements $i$, $j$, and $k$ of $\{1, 2, 3\}$ such that $C_j - C_i \neq C_k - C_i$.*

(xi) *If $C_1$ and $C_2$ form a modular pair of intersecting circuits, then $C_1 \triangle C_2$ is a circuit.*

(xii) *If $H_1$ and $H_2$ are distinct hyperplanes, then there is a hyperplane that contains $H_1 \cap H_2$ and is contained in $H_1 \cup H_2$.*

(xiii) *If $B$ is a basis and $C$ is a circuit, then $C = \triangle_{e \in C-B} C(e, B)$.*

(xiv) *There is a basis $B$ of $M$ such that if $C$ is a circuit, then $C = \triangle_{e \in C-B} C(e, B)$.*

(xv) *If* $B_1$ *and* $B_2$ *are bases of* $M$ *and* $f \in B_2$, *then the number of elements* $e$ *of* $B_1$ *such that both* $(B_1 - e) \cup f$ *and* $(B_2 - f) \cup e$ *are bases is odd.* $\square$

## 5.2. The main theorem

For a minor-closed class of matroids $\mathcal{N}$, let $\mathcal{N}^-$ be the set of excluded-minors for $\mathcal{N}$. Recall that, $\mathcal{M}$ is any class of matroids closed under isomorphisms, direct-sums, and the taking of minors. $\mathcal{M}_1$ is the class of matroids with the property that a matroid is in $\mathcal{M}_1$ if its restriction to every hyperplane is in $\mathcal{M}$. Note that $\mathcal{M}_1$ is a minor-closed class. To see this, it is sufficient to show that if $M$ is in $\mathcal{M}_1$, then for all $e \in E(M)$, both $M/e$ and $M\backslash e$ are in $\mathcal{M}_1$. Now, $\mathcal{H}(M/e)$ is the set of all $X$ in $E(M) - e$ such that $X \cup e$ is a hyperplane of $M$ and $\mathcal{H}(M\backslash e)$ is the set of maximal proper subsets of $E(M) - e$ of the form $H - e$ where $H$ is a hyperplane of $M$. Therefore, if the restriction of $M$ to every hyperplane is binary, then both $M/e$ and $M\backslash e$ have the property that every restriction to one of their respective hyperplanes is binary.

**Theorem 5.2.1.** *Assume that* $U_{0,1}$ *is in* $\mathcal{M}$. *The following statements are equivalent for a loopless matroid* $M$:

(i) $M$ *is in* $\mathcal{M}_1$.

(ii) $M$ *has no* $U_{1,1} \oplus N$*-minor for all* $N$ *in* $\mathcal{M}^-$.

(iii) *Every disconnected series-minor of* $M$ *is in* $\mathcal{M}$.

(iv) *Every disconnected restriction of* $M$ *is in* $\mathcal{M}$.

**Proof.** Suppose that (i) holds and consider $U_{1,1} \oplus N$ where $N$ is in $\mathcal{M}^-$. As $E(N)$ is a hyperplane of $U_{1,1} \oplus N$, the matroid $U_{1,1} \oplus N$ is not in $\mathcal{M}_1$. Since $M$ is in $\mathcal{M}_1$ and $\mathcal{M}_1$ is closed under minors, $M$ cannot have a minor isomorphic to $U_{1,1} \oplus N$ for all $N$ in $\mathcal{M}^-$. Hence (i) implies (ii).

Next, suppose that (ii) holds and assume that $M$ has a disconnected series-minor $Z$ that is not in $\mathcal{M}$. Then $Z = M\backslash X/Y$ for some sets $X$ and $Y$, such that every element of $Y$ is in series with an element of $M\backslash X$ not in $Y$. Now $Z = Z_1 \oplus Z_2$, and without loss of generality, we may assume that $Z_2 \notin \mathcal{M}$. If $r(Z_1) \geq 1$, then, for some $N$ in $\mathcal{M}^-$, the matroid $Z$ has a $U_{1,1} \oplus N$-minor, a contradiction to (ii). Thus we may assume that $r(Z_1) = 0$, and hence that $Z$ has a loop, say $e$. Since $M$ has no loops, it follows that there is an element $f$ of $Y$ such that $M\backslash X/(Y - f)$ has $\{e, f\}$ as a circuit. But $f$ is in series with some element $g$ of $M\backslash X$ that is not in $Y$. Thus $\{f, g\}$ is a cocircuit of $M\backslash X$, and hence of $M\backslash X/(Y - f)$. Since this cocircuit cannot meet the circuit $\{e, f\}$ in a single element, we conclude that $g = e$. Hence $\{e, f\}$ is both a circuit and a cocircuit of $M\backslash X/(Y - f)$, so $\{e, f\}$ is a component of this matroid. Thus $M\backslash X/(Y - f)$ is the direct-sum of $U_{1,2} \oplus Z_2$ and therefore $M$ has a $U_{1,1} \oplus N$-minor for some $N$ in $\mathcal{M}^-$. This contradiction to (ii) completes the proof that (ii) implies (iii).

Clearly (iii) implies (iv). It remains to show that (iv) implies (i). Suppose that (iv) holds. Let $H$ be a hyperplane of $M$ and $e \in E(M) - H$. The matroid $M|(H \cup e)$ has $e$ as a coloop, and is therefore a disconnected restriction of $M$. By assumption, $M|(H \cup e)$ is in $\mathcal{M}$, and hence, so is $M|H$. This completes the proof that (iv) implies (i) and thereby finishes the proof of the theorem. $\square$

To see that one cannot replace "series" by "parallel" in (iii), let $\mathcal{M}$ be the class of binary matroids and consider the matroid $M_1$ shown in Figure 29. Every disconnected parallel minor of $M$ is in $\mathcal{M}$, but clearly the set $\{1, 2, 3, 4\}$ is a non-binary hyperplane. It is also interesting to note that the statement "Every disconnected minor of $M$ is in $\mathcal{M}$" is not included in the statement of Theorem 5.2.1. In fact this statement is much stronger than the statements (i)-(iv) in the theorem. To see this, let $\mathcal{M}$ be the class of binary matroids and consider the matroid $M_2$ shown in Figure 29. Every disconnected restriction of $M_2$ is binary. However, $M$ has a disconnected minor that is non-binary.

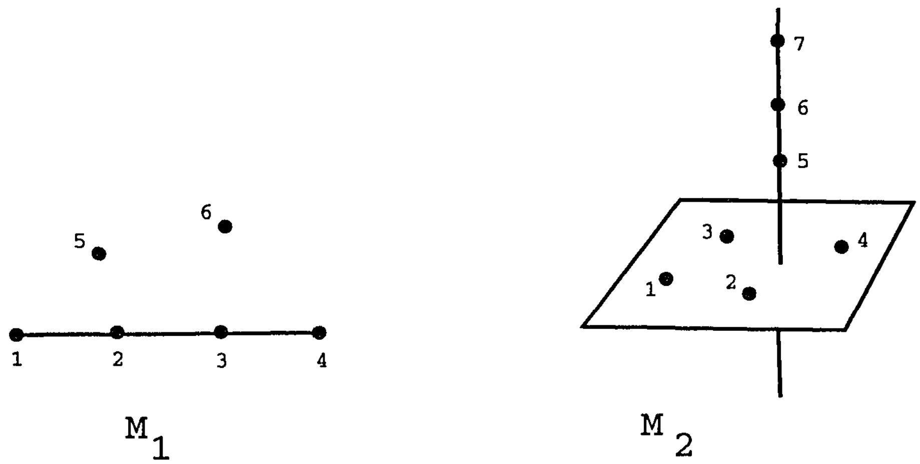



**Figure 29.** $M_1$ and $M_2$

## 5.3. The matroids in which all hyperplanes are binary

In this section we will concentrate on the case when $\mathcal{M}$ is binary. Then $\mathcal{M}_1$ is the class of matroids whose restriction to every hyperplane is binary. Members of $\mathcal{M}_1$ include all binary matroids, all uniform matroids, all rank-3 matroids with no $k$-point lines for $k \geq 4$, and all relaxations of binary matroids. Using the fact that $U_{2,4}$ is the only excluded minor for the class of binary matroids, it follows from Theorem 5.2.1 that $M|H$ is binary for every hyperplane $H$ if and only if $M$ has no $(U_{1,1} \oplus U_{2,4})$-minor. Theorem 5.1.1 lists several well-known characterizations for the class of binary matroids. In most cases one can obtain similar characterizations for matroids whose restriction to every hyperplane is binary. The following two propositions, the first of which is straightforward, contain several such characterizations.

**Proposition 5.3.1.** *The following statements are equivalent for a matroid $M$:*

(i) *$M|H$ is binary for all hyperplanes $H$ of $M$.*

(ii) *If $C_1$ and $C_2$ are distinct circuits such that $C_1 \cup C_2$ is non-spanning, then $C_1 \triangle C_2$ is a disjoint union of circuits.*

(iii) *If $C_1$ and $C_2$ are distinct circuits such that $C_1 \cup C_2$ is non-spanning, then $C_1 \triangle C_2$ contains a circuit.*

(iv) *The symmetric difference of any set of circuits whose union is non-spanning is a disjoint union of circuits.*

(v) *The symmetric difference of any set of circuits whose union is non-spanning is either empty or contains a circuit.*

(vi) *If $C_1$ and $C_2$ are distinct circuits such that $C_1 \cup C_2$ is non-spanning, and $e$ and $f$ are elements of $C_1 \cup C_2$, then $(C_1 \cup C_2) - \{e, f\}$ contains a circuit.*

(vii) *If $C_1$, $C_2$, and $C_3$ are distinct circuits such that $C_1 \cup C_2 \cup C_3$ is non-spanning and $C_1 \cap C_2 \cap C_3 \neq \emptyset$, then there are distinct element $i$, $j$, $k$ of $\{1, 2, 3\}$ such that $C_j - C_i \neq C_k - C_i$.*

(viii) *If $C_1$ and $C_2$ form a modular pair of intersecting circuits such that $C_1 \cup C_2$ is non-spanning, then $C_1 \triangle C_2$ is a circuit.*

**Proof.** The proof of (i)-(viii) is a direct consequence of Theorem 5.1.1 since a non-spanning set is contained in a hyperplane.

Notice that if 5.1.1(xiv) is modified by requiring that $C$ is a non-spanning circuit, the resulting condition is not equivalent to (i)-(viii) above. To see this, consider the identically sel-dual matroid $M_3$ in Figure 30. $M_3$ restricted to every hyperplane is binary. Consider the basis $B = \{1, 2, 3\}$ and non-spanning circuit $C = \{3, 5, 6\}$. Then $C(5, B) = \{1, 2, 3, 5\}$ and $C(6, B) = \{1, 2, 3, 6\}$, but $C(5, B) \triangle C(6, B) = \{5, 6\}$. Clearly $\{5, 6\}$ is not a circuit. Observe that all bases in $M_3$ look alike. Hence $M_3$ has no basis $B$ such that if $C$ is a non-spanning circuit, then $C = \triangle_{e \in C-B} C(e, B)$.

The next proposition characterizes the class $\mathcal{M}_1$ in terms of the cardinality of the intersections of circuits and cocircuits. The characterization is not as obvious as those given above. To see this consider again the matroid $M_3$ in Figure 30. $M_3$ is in $\mathcal{M}_1$. However, the set $\{1, 2, 4\}$ is a non-spanning and non-cospanning circuit and a non-spanning and non-cospanning cocircuit.

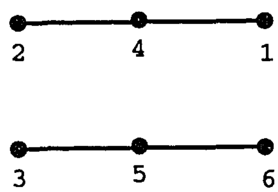



**Figure 30.** $M_3$

**Proposition 5.3.2.** *The following statements are equivalent for a matroid $M$:*

(i) *$M$ is in $\mathcal{M}_1$.*

(ii) *If $C$ is a circuit of $M$ and $C_1^*$ and $C_2^*$ are a modular pair of circuits of $M^*$ such that $C \cap C_1^* = \emptyset$, then $|C \cap C_2^*|$ is even.*

(iii) *If $C$ is a circuit of $M$ and $C_1^*$ and $C_2^*$ are a modular pair of circuits of $M^*$ such that $C \cap C_1^* = \emptyset$, then $|C \cap C_2^*| \neq 3$.*

**Proof.** Suppose that (i) holds. Let $C$ be a circuit and $C_1^*$ and $C_2^*$ be a modular pair of circuits of $M^*$ such that $C \cap C_1^* = \phi$. Then $C$ is a circuit of $M \backslash C_1^*$. Since $E(M) - C_1^*$ is a hyperplane, $M \backslash C_1^*$ is a binary matroid. Certainly $C_2^* - C_1^*$ contains a cocircuit of $M \backslash C_1^*$. Since $C_1^*$ and $C_2^*$ form a modular pair of cocircuits, $r^*(C_1^* \cup C_2^*) = |C_1^* \cup C_2^*| - 2$. Therefore $r(C_1^* \cup C_2^*) = 2$ and $r(E(M) - (C_1^* \cup C_2^*)) = r(M) - 2$. This implies that $E(M) - (C_1^* \cup C_2^*)$ is a hyperplane of $M \backslash C_1^*$ and therefore, $C_2^* - C_1^*$ is a cocircuit of $M \backslash C_1^*$. Therefore, by 5.1.1(ii) $|C \cap (C_2^* - C_1^*)| = |C \cap C_2^*|$ is even.

Conversely, suppose (ii) holds and assume that $M|H$ is non-binary for some hyperplane $H$. Then by 5.1.1(ii) $M|H$ has a cocircuit $D^*$ and a circuit $C$ such that $|C \cap D^*|$ is odd. Let $C^* = E(M) - H$. Then $C^*$ is a cocircuit of $M$. Now $M|H$, which equals $M\backslash C^*$, has $D^*$ as a cocircuit. This implies that $D^* \cup C_1^*$ is a cocircuit for some $C_1^* \subseteq C^*$. Now $H - D^*$ is a hyperplane of $M|H$, and $r(H - D^*) = r(H) - 1 = r(M) - 2$. Since $H - D^* = E(M) - (C^* \cup D^*)$, therefore $r(E(M) - (C^* \cup D^*)) = r(M) - 2$, and $r(C^* \cup D^*) = 2$. This implies that $r(C^* \cup D^*) = |C^* \cup D^*| - 2$ and therefore, $D^* \cup C_1^*$ and $C^*$ form a modular pair of cocircuits of $M$ such that $C \cap C^* = \emptyset$. However $|C \cap (D^* \cup C_1^*)| = |C \cap D^*|$ is odd. This contradiction to (ii) completes the proof that (ii) implies (i).

The arguments used above that established the equivalence of (i) and (ii) can also be used to establish the equivalence of (i) and (iii).

## 5.4. A generalization of the main theorem

For $k \geq 1$, let $\mathcal{M}_k$ be the class of matroids with the property that a matroid $M$ is in $\mathcal{M}_k$ if its restriction to every flat of rank $r(M) - k$ is in $\mathcal{M}$.

**Proposition 5.4.1.** *For $k \geq 1$, $M$ is in $\mathcal{M}_k$ if and only if $M$ has no minor isomorphic to $U_{k,k} \oplus N$ for all $N$ in $\mathcal{M}^-$.*

**Proof.** We shall first show that the class $\mathcal{M}_k$ is closed under minors. To see this, it is sufficient to show that if $M$ is in $\mathcal{M}_k$ then, for all $e \in E(M)$, both $M/e$ and $M\backslash e$ are in $\mathcal{M}_k$. The sets of flats of $M\backslash e$ are the sets $F - e$ in $E(M) - e$ such that $F$ is a flat of $M$. If the deletion of $e$ does not effect the rank of $M$ then $r(M\backslash e) = r(M)$ and $r(F - e) = r(F) = r(M) - k$. If the deletion of $e$ does effect the rank then $r(M\backslash e) = r(M) - 1$, and $r(F - e) = r(F) - 1 = r(M) - k - 1 = r(M\backslash e) + 1 - k - 1 = r(M\backslash e) - k$. So, in any case, if $F$ is a flat of $M$ of rank $r(M)-k$ then $F-e$ is a flat of $M\backslash e$ of rank $r(M\backslash e)-k$. Using the fact that $r(M/e)=r(M)-1$, observe that, the sets of flats of rank $r(M/e)-k$ of $M/e$ are the sets $X$ in $E(M)-e$ such that $X\cup e$ is a flat of $M$ of rank $r(M)-k$. Therefore, if $M$ restricted to every flat of rank $r(M)-k$ is binary, then so are $M/e$ restricted to every flat of rank $r(M/e)-k$ of $M/e$, and $M\backslash e$ restricted to every flat of rank $r(M\backslash e)-k$ of $M\backslash e$. Next consider the matroid $U_{k,k}\oplus N$ where $N\in\mathcal{M}^-$. This matroid has $E(N)$ as a flat of rank $r(U_{k,k}\oplus N)-k$ and therefore, it is not in $\mathcal{M}_k$. Since $M$ is in $\mathcal{M}_k$ and $\mathcal{M}_k$ is closed under minors, $M$ cannot have a minor isomorphic to $U_{k,k}\oplus N$.

Conversely, suppose that $M$ is a matroid such that for all $N$ in $\mathcal{M}^-$, $M$ has no minor isomorphic to $U_{k,k}\oplus N$. Suppose, for some flat $F$ of $M$ of rank $r(M)-k$, the matroid $M|F\notin\mathcal{M}$. Then $F$ has a minor isomorphic to $N$, for some $N\in\mathcal{M}^-$. Since $F$ is a flat of rank $r(M)-k$, if $B_F$ is a basis of $F$, then there are elements $e_1,e_2\dots,e_k$ of $E(M)-F$, such that $B_F\cup\{e_1,e_2\dots,e_k\}$ is a basis of $M$. Then $M|(F\cup\{e_1,\dots e_k\})\cong M|(F\oplus U_{k,k})$. As $M|F$ has an $N$-minor, it follows that $M|(F\cup\{e_1,e_2,\dots e_k\})$ has a $U_{k,k}\oplus N$-minor. This contradiction to the assumption completes the proof. $\square$

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## APPENDIX

We now give the detailed case-checking postponed from previous chapters. The matroids concerned are all binary and the single-element extensions computed are all 3-connected. Seymour (1980) has computed the binary single-element extensions of the matroids $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, $M^*(K_5)$, and $R_{10}$. We will start by briefly describing the method used here for finding the binary, 3-connected, single-element extensions of a matroid. Suppose $M$ is an $n$-element, rank-$r$ matroid represented over $GF(2)$ by the matrix $A$. Each of the $2^r - (n+1)$ columns $\{x_1, x_2, \ldots, x_{2^r-(n+1)}\}$, when adjoined to the matrix $A$, gives a single-element extension represented by the matrix $A \cup x_i$. Note that, since the extensions are 3-connected, we cannot add the column of zeros or any column already present in the matrix, as these would result in a loop or two parallel elements, respectively. It remains to check whether $M(A \cup x_i) \cong M(A \cup x_j)$ for some $i, j$ in $\{1, 2, \ldots, 2^r - (n+1)\}$. For each matroid $M(A \cup x_i)$, the circuits and cocircuits are computed. Then equivalence classes are created among these extensions based on whether the numbers of $s$-element circuits and $t$-element cocircuits match up for all $s$ in $\{2, 3, \ldots, rank + 1\}$ and all $t$ in $\{2, 3, \ldots, corank + 1\}$. Clearly two isomorphic extensions must be in the same equivalence class. Although we cannot prove, in general, that two members of the same equivalence class are isomorphic, in each of our examples we are able to find such isomorphisms. These isomorphisms are found with the help of the automorphisms of $M$.

For example, consider the matroid $P_9$. Table 1a lists the numbers of circuits and cocircuits, by size, of each extension of $P_9$. Table 1b lists the columns that can be adjoined to the matrix representing $P_9$, to obtain each of $(P_9, ext1)$,

$(P_9, ext2)$, and $(P_9, ext3)$. Then some automorphisms of $P_9$ are computed and expressed in terms of the row operations that induce them. In Figure A-1 these automorphisms are used to show that the matrix obtained by adjoining each of the columns (1010), (0110), (1001), or (0101) result in isomorphic matroids. Therefore any one of the following matrices is a valid representation for $(P_9, ext1)$.

$$\begin{array}{c} \begin{array}{cccc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \end{array} \qquad \begin{array}{c} \begin{array}{cccc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \end{array}$$

$$\begin{array}{c} \begin{array}{cccc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right) \end{array} \qquad \begin{array}{c} \begin{array}{cccc|cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right) \end{array}$$

Note that, the first matrix is obtained by adjoining the column (1010) which is written in bold in Table 1b. This is the representation of $(P_9, ext1)$ used in calculations. In general, if there is more than one extension column that can be adjoined to the matrix to obtain a particular extension, then the extension column written in bold should be used. If no column is highlighted then any one of the listed columns may be used to form the extension.

**The matroid $\mathbf{P_9}$**

$$
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left(\begin{array}{cccc|ccccc}
 & & & & 0 & 1 & 1 & 1 & 1 \\
 & & I_4 & & 1 & 0 & 1 & 1 & 1 \\
 & & & & 1 & 1 & 0 & 1 & 0 \\
 & & & & 1 & 1 & 1 & 1 & 0
\end{array}\right)
\end{array}
$$

**Table 1a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(P_9, ext1)$ | Circuits | 0 | 9 | 16 | 15 | | |
| | Cocircuits | 0 | 0 | 3 | 6 | 4 | 2 |
| $(P_9, ext2)$ | Circuits | 0 | 8 | 18 | 16 | | |
| | Cocircuits | 0 | 0 | 2 | 8 | 4 | 0 |
| $(P_9, ext3)$ | Circuits | 0 | 10 | 16 | 12 | | |
| | Cocircuits | 0 | 1 | 1 | 6 | 6 | 0 |

**Table 1b. Single-element extensions of $\mathbf{P_9}$**

| $(P_9, ext1)$ | $(P_9, ext2)$ | $(P_9, ext3)$ |
|---|---|---|
| **(1010)** | (1110) | (0011) |
| (0110) | | |
| (1001) | | |
| (0101) | | |

**Automorphisms of $\mathbf{P_9}$**

Swapping row 1 with row 2 induces an automorphism on $P_9$. Pivoting on element $[a_{1,6}]$ and swapping row 3 with row 4 also induces an automorphism on $P_9$. The corresponding maps on $(x_1,\ x_2,\ x_3\ ,x_4)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3\ ,x_4)^T \longrightarrow (x_2,\ x_1,\ x_3,\ x_4)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4)^T \longrightarrow (x_1,\ x_2,\ x_4 + x_1,\ x_3 + x_1)^T$$

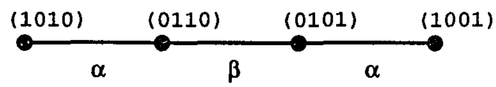


**Figure A-1**

**The matroid $\mathbf{P_9^*}$**

$$\begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$\left(\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 \end{array}\right)$$

**Table 2a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $(P_9^*, ext1)$ | Circuits | 0 | 4 | 6 | 8 | 6 |
| | Cocircuits | 0 | 4 | 6 | 8 | 6 |
| $(P_9^*, ext2)$ | Circuits | 0 | 2 | 8 | 12 | 4 |
| | Cocircuits | 0 | 2 | 8 | 12 | 4 |
| $(P_9^*, ext3)$ | Circuits | 0 | 1 | 10 | 11 | 4 |
| | Cocircuits | 0 | 1 | 10 | 11 | 4 |
| $(P_9^*, ext4)$ | Circuits | 0 | 3 | 7 | 10 | 5 |
| | Cocircuits | 0 | 3 | 7 | 10 | 5 |
| $(P_9^*, ext5)$ | Circuits | 0 | 2 | 9 | 9 | 6 |
| | Cocircuits | 0 | 2 | 9 | 9 | 6 |
| $(P_9^*, ext6)$ | Circuits | 0 | 3 | 6 | 11 | 6 |
| | Cocircuits | 0 | 2 | 10 | 8 | 4 |
| $(P_9^*, ext7)$ | Circuits | 0 | 2 | 7 | 12 | 6 |
| | Cocircuits | 0 | 0 | 16 | 0 | 12 |
| $(P_9^*, ext8)$ | Circuits | 0 | 2 | 10 | 8 | 4 |
| | Cocircuits | 0 | 3 | 6 | 11 | 6 |

**Table 2b. Single-element extensions of $\mathbf{P_9^*}$**

| $(P_9^*, ext1)$ | $(P_9^*, ext2)$ | $(P_9^*, ext3)$ | $(P_9^*, ext4)$ |
|---|---|---|---|
| **(11000)** | (11100) | (11001) | (10010) |
| (11111) | **(11011)** | **(11101)** | (01010) |
| | | | **(10110)** |
| | | | (01110) |
| | | | (10001) |
| | | | (01001) |
| | | | (10101) |
| | | | (01101) |

| $(P_9^*, ext5)$ | $(P_9^*, ext6)$ | $(P_9^*, ext7)$ | $(P_9^*, ext8)$ |
|---|---|---|---|
| **(10100)** | (00111) | (00011) | **(00110)** |
| (01100) | | | (00101) |
| (10011) | | | |
| (01011) | | | |

### Automorphisms of $\mathbf{P_9^*}$

Swapping rows 1 and 2 in the matrix representing $P_9^*$ induces an automorphism on $P_9^*$. Pivoting on element $[a_{2,6}]$ and then swapping rows 4 and 5 also induces an automorphism on $P_9^*$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_2,\ x_1,\ x_3,\ x_4,\ x_5)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1,\ x_2,\ x_3 + x_2,\ x_5 + x_2,\ x_4 + x_2)^T$$

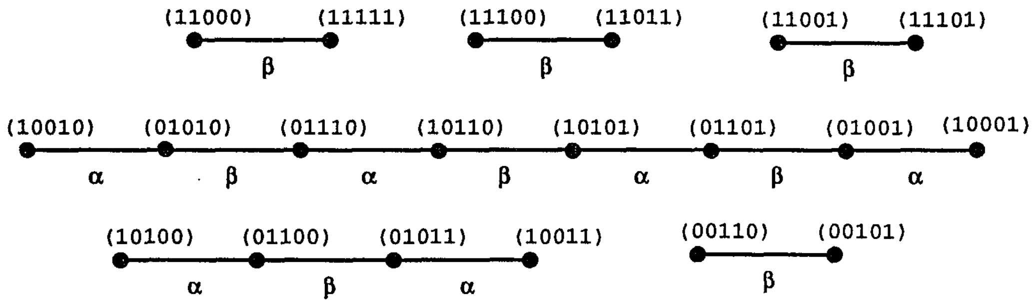


**Figure A-2**

The permutation on $\{1,2,\ldots,10\}$ that maps $(1,2,\ldots,10)$ to $(3,10,7,8,1,9,5,4,2,6)$ is an isomorphism from $(P_9^*, ext2)$ to $(P_9, ext2)^*$. The permutation that maps $(1,2,\ldots,10)$ to $(6,7,9,10,8,1,2,5,3,4)$ is an isomorphism from $(P_9^*, ext3)$ to $(P_9, ext3)^*$. The permutation that maps $(1,2,\ldots,10)$ to $(7,5,3,4,6,2,1,8,9,10)$ is an isomorphism from $(P_9^*, ext4)$ to $(P_9, ext4)^*$. The permutation that maps $(1,2,\ldots,10)$ to $(6,8,7,9,10,1,3,2,4,5)$ is an isomorphism from $(P_9^*, ext5)$ to $(P_9, ext5)^*$. The permutation that maps $(1,2,\ldots,10)$ to $(1,6,3,10,9,7,2,4,8,5)$ is an isomorphism from $(P_9^*, ext8)$ to $(P_9, ext6)^*$. Therefore, each of $(P_9^*, ext1)$, $(P_9^*, ext2)$, $(P_9^*, ext3)$, $(P_9^*, ext4)$, and $(P_9^*, ext5)$ is self-dual, and $(P_9^*, ext8) \cong (P_9^*, ext6)^*$.

**The matroid $M(K_{3,3})$**

$$
\begin{array}{ccccc}
1\ 2\ 3\ 4\ 5 & 6 & 7 & 8 & 9 \\
\end{array}
$$
$$
\left(\begin{array}{c|cccc}
 & 1 & 0 & 0 & 1 \\
 & 1 & 1 & 0 & 1 \\
I_5 & 1 & 1 & 1 & 1 \\
 & 0 & 1 & 1 & 1 \\
 & 0 & 0 & 1 & 1
\end{array}\right)
$$

**Table 3a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $(K_{3,3}, ext1)$ | Circuits | 0 | 3 | 9 | 6 | 6 |
| | Cocircuits | 0 | 4 | 5 | 9 | 7 |
| $(K_{3,3}, ext2)$ | Circuits | 0 | 2 | 9 | 9 | 6 |
| | Cocircuits | 0 | 2 | 9 | 9 | 6 |
| $(K_{3,3}, ext3)$ | Circuits | 0 | 0 | 16 | 0 | 12 |
| | Cocircuits | 0 | 2 | 7 | 12 | 6 |
| $(K_{3,3}, ext4)$ | Circuits | 0 | 0 | 15 | 0 | 15 |
| | Cocircuits | 0 | 0 | 15 | 0 | 15 |

**Table 3b. Single-element extensions of $M(K_{3,3})$**

| $(K_{3,3}, ext1)$ | $(K_{3,3}, ext2)$ | $(K_{3,3}, ext3)$ | $(K_{3,3}, ext4)$ |
|---|---|---|---|
| **(11000)** | (10100) | **(11010)** | (10101) |
| (01100) | **(10010)** | (10110) | |
| (00110) | (01010) | (11001) | |
| (11110) | (10001) | (01101) | |
| (00011) | (01001) | (10011) | |
| (01111) | (00101) | (01011) | |
| | (11101) | | |
| | (11011) | | |
| | (10111) | | |

**Automorphisms of $M(K_{3,3})$**

Pivoting on element $[a_{1,9}]$ and then swapping row 2 with row 5 and row 3 with row 4 induces an automorphism on $M(K_{3,3})$. Pivoting on element $[a_{2,9}]$ and then swapping row 3 with row 5 induces an automorphism on $M(K_{3,3})$. Pivoting first on element $[a_{1,6}]$, then on element $[a_{5,8}]$, and finally on element $[a_{3,7}]$, and then swapping row 2 with row 4 also induces an automorphism on $M(K_{3,3})$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1,\ x_5 + x_1,\ x_4 + x_1,\ x_3 + x_1,\ x_2 + x_1)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1 + x_2,\ x_2,\ x_5 + x_2,\ x_4 + x_2,\ x_3 + x_2)^T$$

$$\gamma : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1,\ x_1 + x_3 + x_4,\ x_1 + x_3 + x_5,\ x_2 + x_3 + x_5,\ x_5)^T$$

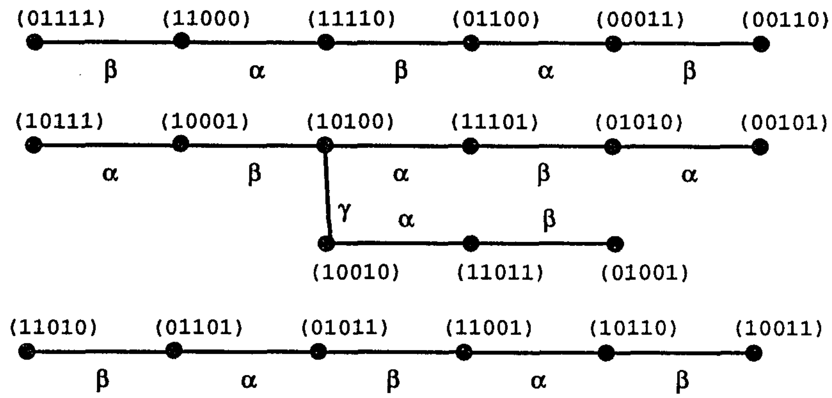


**Figure A-3**

Observe that the matroid $(K_{3,3}, ext1) \cong K'_{3,3}$. The permutation of $\{1, 2, \ldots, 10\}$ that maps $(1, 2, \ldots, 10)$ to $(1, 2, 10, 3, 5, 6, 9, 7, 8, 4)$ is an isomorphism from $(K_{3,3}, ext2)$ to $(P_9^*, ext5)$. The permutation that maps $(1, 2, \ldots, 10)$ to $(3, 4, 1, 2, 5, 10, 6, 7, 9, 8)$ is an isomorphism from $(K_{3,3}, ext3)$ to $(P_9^*, ext7)^*$. The permutation that maps $(1, \ldots, 10)$ to $(6, 2, 3, 4, 5, 10, 1, 7, 8, 9)$ is an isomorphism from $(K_{3,3}, ext4)$ to $R_{10}$. Therefore $(K_{3,3}, ext2) \cong (P_9^*, ext5)$, $(K_{3,3}, ext3) \cong (P_9^*, ext7)^*$, and $(K_{3,3}, ext4) \cong R_{10}$

**The matroid $M^*(K_{3,3})$**

$$\begin{array}{c} \begin{array}{cccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{c|ccccc} & 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 \\ I_4 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array}$$

**Table 4a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(K^*_{3,3}, ext1)$ | Circuits | 0 | 9 | 16 | 15 | | |
| | Cocircuits | 0 | 0 | 3 | 6 | 4 | 2 |

**Table 4b. Single-element extensions of $M^*(K_{3,3})$**

| $(K^*_{3,3}, ext1)$ |
|---|
| (1100) |
| (1010) |
| (0110) |
| **(1110)** |
| (0101) |
| (1011) |

**Automorphisms of $M^*(K_{3,3})$**

Pivoting on element $[a_{1,7}]$ and swapping row 3 with row 4 induces an automorphism on $M^*(K_{3,3})$. Pivoting on element $[a_{4,9}]$ and swapping row 1 with row 2 also induces an automorphism on $M^*(K_{3,3})$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3\ ,x_4)^T \longrightarrow (x_1,\ x_2 + x_1,\ x_4 + x_1,\ x_3 + x_1)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4)^T \longrightarrow (x_2,\ x_1,\ x_3 + x_4,\ x_4)^T$$

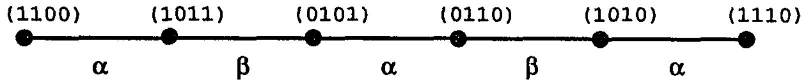


**Figure A-4**

Observe that the permutation of $\{1, 2, \ldots, 10\}$ that maps $(1, 2, \ldots, 10)$ to $(4, 1, 3, 2, 10, 8, 6, 7, 5, 9)$ is an isomorphism from $(K^*_{3,3}, ext1)$ to $(P_9, ext1)$. Therefore $(K^*_{3,3}, ext1) \cong (P_9, ext1)$

**The matroid $E_5$**

$$\begin{array}{c} \begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \end{array}\right) \end{array}$$

**Table 5a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(E_5, ext1)$ | Circuits | 0 | 5 | 11 | 15 | 11 | |
| | Cocircuits | 0 | 1 | 6 | 7 | 8 | 5 |
| $(E_5, ext2)$ | Circuits | 0 | 3 | 16 | 13 | 12 | |
| | Cocircuits | 0 | 1 | 4 | 10 | 9 | 4 |
| $(E_5, ext3)$ | Circuits | 0 | 4 | 12 | 18 | 10 | |
| | Cocircuits | 0 | 1 | 4 | 11 | 8 | 3 |
| $(E_5, ext4)$ | Circuits | 0 | 5 | 12 | 15 | 10 | |
| | Cocircuits | 0 | 2 | 3 | 9 | 10 | 2 |
| $(E_5, ext5)$ | Circuits | 0 | 4 | 11 | 18 | 12 | |
| | Cocircuits | 0 | 0 | 7 | 9 | 6 | 6 |
| $(E_5, ext6)$ | Circuits | 0 | 2 | 16 | 16 | 12 | |
| | Cocircuits | 0 | 0 | 5 | 12 | 7 | 4 |
| $(E_5, ext7)$ | Circuits | 0 | 3 | 15 | 13 | 15 | |
| | Cocircuits | 0 | 0 | 7 | 8 | 7 | 8 |

**Table 5b.Single-element extensions of M($E_5$)**

| $(E_5, ext1)$ | $(E_5, ext2)$ | $(E_5, ext3)$ | $(E_5, ext4)$ |
|---|---|---|---|
| (11000) | (01100) | (11100) | (01010) |
| (10110) | (00110) | (10010) | (01110) |
| (10101) | (00101) | (10001) | |
| (11111) | (01011) | (11011) | |

| $(E_5, ext5)$ | $(E_5, ext6)$ | $(E_5, ext7)$ |
|---|---|---|
| (01001) | (11001) | (10011) |
| (01101) | (11101) | |
| (00011) | | |
| (00111) | | |

**Automorphisms of M($E_5$)**

Pivoting on element $[a_{2,6}]$ and then swapping row 4 with row 5 induces an automorphism on $M(E_5)$. Pivoting first on element $[a_{1,7}]$ and then on element $[a_{1,8}]$ and swapping row 2 with row 4 induces an automorphism on $M(E_5)$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1,\ x_2,\ x_3 + x_2,\ x_5 + x_2,\ x_4 + x_2)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1,\ x_4 + x_1,\ x_3,\ x_2 + x_1,\ x_5)^T$$

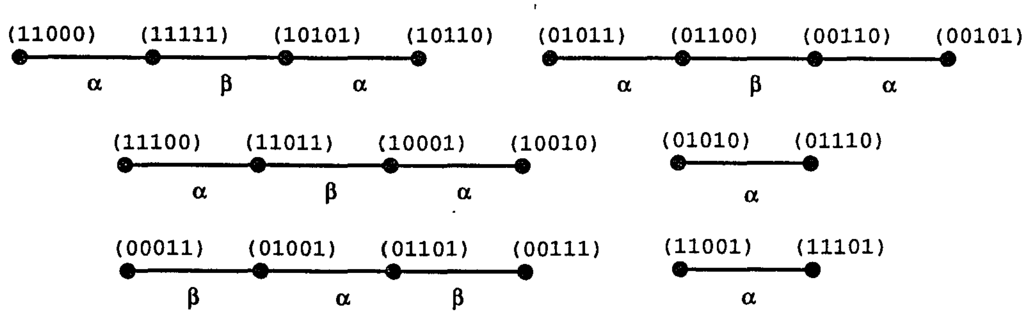


**Figure A-5**

**The matroid $\mathbf{R_{10}}$**

$$
\begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left(\begin{array}{c|ccccc}
 & 1 & 1 & 0 & 0 & 1 \\
 & 1 & 1 & 1 & 0 & 0 \\
I_5 & 0 & 1 & 1 & 1 & 0 \\
 & 0 & 0 & 1 & 1 & 1 \\
 & 1 & 0 & 0 & 1 & 1
\end{array}\right)
\end{array}
$$

**Table 6a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(R_{10}, ext1)$ | Circuits | 0 | 3 | 15 | 13 | 15 | |
| | Cocircuits | 0 | 0 | 7 | 8 | 7 | 8 |
| $(R_{10}, ext2)$ | Circuits | 0 | 0 | 25 | 0 | 27 | |
| | Cocircuits | 0 | 0 | 5 | 10 | 10 | 5 |

**Table 6b. Single-element extensions of $\mathbf{R_{10}}$**

| $(R_{10}ext1)$ | $(R_{10}, ext2)$ |
|---|---|
| (11000) | (11010) |
| (10100) | (10110) |
| (01100) | (10101) |
| (10010) | (01101) |
| (01010) | (01011) |
| (00110) | **(11111)** |
| (11110) | |
| (10001) | |
| (01001) | |
| (00101) | |
| (11101) | |
| (00011) | |
| (11011) | |
| (10111) | |
| **(01111)** | |

## Automorphisms of $\mathbf{R_{10}}$

Cyclically permuting the rows of the above matrix induces an automorphism on $R_{10}$. Pivoting on element $[a_{1,6}]$ and swapping row 3 with row 4 induces an automorphism on $R_{10}$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3,\ x_4)^T \longrightarrow (x_5,\ x_1,\ x_2,\ x_3,\ x_4)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4)^T \longrightarrow (x_1,\ x_2 + x_1,\ x_4,\ x_3,\ x_5 + x_1)^T$$

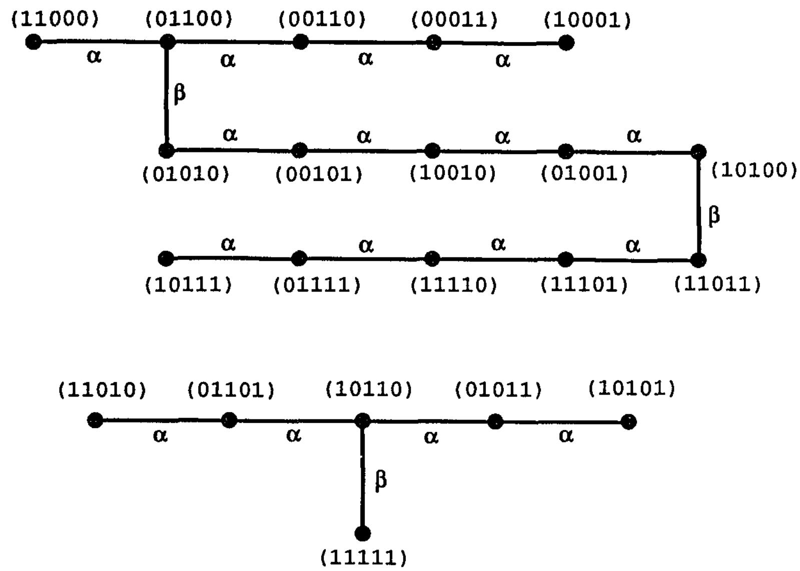


**Figure A-6**

Observe that the permutation of $\{1,2,\ldots,11\}$ that maps $(1,2,\ldots,11)$ to $(5,3,11,4,1,6,7,2,8,10)$ is an isomorphism from $(R_{10}, ext1)$ to $(E_5, ext7)$. Therefore $(R_{10}, ext1) \cong (E_5, ext7)$

**The matroid $\mathbf{M(A_5)\backslash a_6}$**

$$\begin{array}{c} \begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{ccccc|ccccc} & & & & & 0 & 1 & 1 & 1 & 1 \\ & & & & & 1 & 0 & 1 & 1 & 1 \\ & & I_5 & & & 1 & 1 & 0 & 1 & 0 \\ & & & & & 1 & 1 & 1 & 1 & 0 \\ & & & & & 1 & 1 & 0 & 0 & 0 \end{array}\right) \end{array}$$

**Table 7a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $(A_5\backslash a_6, ext1)$ | Circuits | 0 | 6 | 10 | 14 | 12 | | |
| | Cocircuits | 0 | 2 | 4 | 8 | 8 | 4 | |
| $(A_5\backslash a_6, ext2)$ | Circuits | 0 | 5 | 11 | 15 | 12 | | |
| | Cocircuits | 0 | 1 | 6 | 7 | 8 | 4 | |
| $(A_5\backslash a_6, ext3)$ | Circuits | 0 | 5 | 10 | 16 | 12 | | |
| | Cocircuits | 0 | 0 | 10 | 0 | 16 | 0 | |
| $(A_5\backslash a_6, ext4)$ | Circuits | 0 | 4 | 12 | 18 | 10 | | |
| | Cocircuits | 0 | 1 | 4 | 11 | 8 | 2 | |
| $(A_5\backslash a_6, ext5)$ | Circuits | 0 | 6 | 10 | 14 | 11 | | |
| | Cocircuits | 0 | 2 | 4 | 8 | 8 | 4 | |
| $(A_5\backslash a_6, ext6)$ | Circuits | 0 | 6 | 10 | 14 | 9 | | |
| | Cocircuits | 0 | 2 | 4 | 8 | 7 | 4 | |
| $(A_5\backslash a_6, ext7)$ | Circuits | 0 | 5 | 11 | 15 | 11 | | |
| | Cocircuits | 0 | 1 | 6 | 7 | 8 | 5 | |
| $(A_5\backslash a_6, ext8)$ | Circuits | 0 | 6 | 10 | 14 | 10 | | |
| | Cocircuits | 0 | 2 | 4 | 8 | 8 | 2 | |

**Table 7a. Sizes of circuits and cocircuits (continued)**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $(A_5\backslash a_6, ext9)$ | Circuits | 0 | 7 | 9 | 11 | 10 | | |
| | Cocircuits | 0 | 2 | 6 | 4 | 8 | 4 | |
| $(A_5\backslash a_6, ext10)$ | Circuits | 0 | 5 | 11 | 15 | 10 | | |
| | Cocircuits | 0 | 1 | 6 | 7 | 8 | 4 | |
| $(A_5\backslash a_6, ext11)$ | Circuits | 0 | 6 | 12 | 12 | 8 | | |
| | Cocircuits | 0 | 3 | 2 | 7 | 10 | 4 | |
| $(A_5\backslash a_6, ext12)$ | Circuits | 0 | 5 | 9 | 17 | 14 | | |
| | Cocircuits | 0 | 8 | 8 | 4 | 8 | | |

**Table 7b. Single-element extensions of $\mathbf{A_5\backslash a_6}$**

| $(A_6\backslash a_6, ext1)$ | $(A_5\backslash a_6, ext2)$ | $(A_5\backslash a_6, ext3)$ | $(A_5\backslash a_6, ext4)$ |
|---|---|---|---|
| (11100) | (11001) | (11011) | (11101) |

| $(A_6\backslash a_6, ext5)$ | $(A_5\backslash a_6, ext6)$ | $(A_5\backslash a_6, ext7)$ | $(A_5\backslash a_6, ext8)$ |
|---|---|---|---|
| (10010) | (10110) | (10100) | (00111) |
| (01010) | (01110) | (01100) | |
| (10101) | (10001) | (10011) | |
| (01101) | (01001) | (01011) | |

| $(A_6\backslash a_6, ext9)$ | $(A_5\backslash a_6, ext10)$ | $(A_5\backslash a_6, ext11)$ | $(A_5\backslash a_6, ext12)$ |
|---|---|---|---|
| (11111) | (00101) | (00110) | (00011) |

**Automorphisms of $\mathbf{M(A_5)\backslash a_6}$**

Swapping row 1 with row 2 induces an automorphism on $M(A_5)\backslash a_6$. Pivoting on element $[a_{2,6}]$ and then on element $[a_{1,7}]$ also induces an automorphism on $M(K_5)$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_2,\ x_1,\ x_3,\ x_4,\ x_5)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1,\ x_2,\ x_3 + x_2 + x_1,\ x_4 + x_2 + x_1,\ x_5 + x_2 + x_1)^T$$

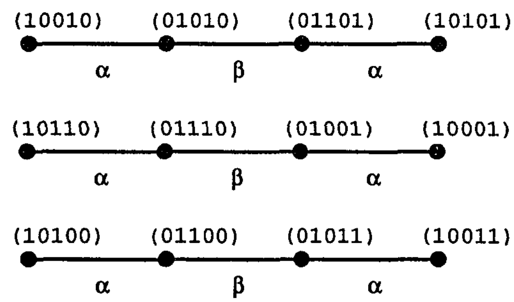


**Figure A-7**

**The matroid $M(K_5)$**

$$\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left(\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 \\ I_4 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \end{array}$$

**Table 8a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(K_5, ext1)$ | Circuits | 0 | 13 | 25 | 25 | | |
| | Cocircuits | 0 | 0 | 1 | 4 | 6 | 4 |

**Table 8b. Single-element extensions of $M(K_5)$**

| $(K_5, ext1)$ |
|---|
| (1010) |
| (1001) |
| (0101) |
| (1101) |
| (1011) |

**Automorphisms of $M(K_5)$**

Pivoting on element $[a_{1,10}]$ and swapping row 2 with row 4 induces an automorphism on $M(K_5)$. Pivoting on element $[a_{4,10}]$ and swapping row 1 with row 3 also induces an automorphism on $M(K_5)$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3,\ x_4)^T \longrightarrow (x_1,\ x_4 + x_1,\ x_3 + x_1,\ x_2 + x_1)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4)^T \longrightarrow (x_3 + x_4,\ x_2 + x_4,\ x_1 + x_4,\ x_4)^T$$

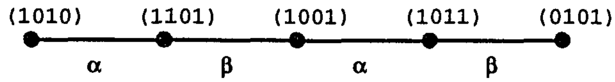


**Figure A-8**

**The matroid $\mathbf{M^*(K_5)}$**

$$\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left(\begin{array}{c|cccc}
 & 1 & 1 & 0 & 0 \\
 & 0 & 1 & 1 & 0 \\
 & 0 & 0 & 1 & 1 \\
I_6 & 1 & 1 & 1 & 0 \\
 & 0 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 1
\end{array}\right)
\end{array}$$

**Table 9a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(K_5^*, ext1)$ | Circuits | 0 | 2 | 5 | 5 | 10 | 4 |
| | Cocircuits | 0 | 6 | 11 | 12 | 10 | |
| $(K_5^*, ext2)$ | Circuits | 0 | 0 | 9 | 0 | 19 | 0 |
| | Cocircuits | 0 | 3 | 16 | 12 | 12 | |
| $(K_5^*, ext3)$ | Circuits | 0 | 0 | 5 | 10 | 10 | 5 |
| | Cocircuits | 0 | 0 | 25 | 0 | 27 | |
| $(K_5^*, ext4)$ | Circuits | 0 | 1 | 5 | 8 | 10 | 4 |
| | Cocircuits | 0 | 4 | 13 | 16 | 9 | |
| $(K_5^*, ext5)$ | Circuits | 0 | 0 | 10 | 0 | 16 | 0 |
| | Cocircuits | 0 | 5 | 10 | 16 | 11 | |

**Table 9b. Single-element extensions of $\mathbf{M^*(K_5)}$**

| $(K_5^*, ext1)$ | $(K_5^*, ext2)$ | $(K_5^*, ext3)$ | $(K_5^*, ext4)$ | $(K_5^*, ext5)$ |
|---|---|---|---|---|
| (101000) | (111000) | (111001) | **(110000)** | (110100) |
| (100100) | (011100) | | (011000) | (101100) |
| (010100) | (110010) | | (001100) | (101010) |
| (010010) | (100110) | | (111100) | (011010) |
| (001010) | (001110) | | (100010) | (010110) |
| (100001) | (110001) | | (111010) | (111110) |
| **(001001)** | **(101001)** | | (000110) | (010101) |
| (000101) | (011001) | | (110110) | **(001101)** |
| (110101) | (111101) | | (101110) | (100011) |
| (000011) | (111011) | | (011110) | (010011) |
| (011011) | | | (010001) | (000111) |
| (100111) | | | (101101) | (101111) |
| (010111) | | | (011101) | |
| (001111) | | | (110011) | |
| (111111) | | | (101011) | |

**Automorphisms of $\mathbf{M^*(K_5)}$**

Swapping row 1 with row 3 and row 4 with row 5 returns a matrix isomorphic to $M^*(K_5)$. Pivoting on element $[a_{1,8}]$ and swapping row 2 with row 4 and row 5 with row 6 induces an automorphism on $M^*(K_5)$. Pivoting on the element $[a_{3,9}]$ and swapping row 2 with row 5 and row 3 with row 6 also induces an automorphism on $M^*(K_5)$. Pivoting on the element $[a_{6,7}]$ and swapping row 1 with row 4 and row 2 with row 3 returns a matrix isomorphic to $M^*(K_5)$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3\ ,x_4,\ x_5,\ x_6)^T \longrightarrow (x_3,\ x_2,\ x_1,\ x_5,\ x_4,\ x_6)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow (x_1,\ x_4 + x_1,\ x_3,\ x_2 + x_1,\ x_6 + x_1,\ x_5 + x_1)^T$$

$$\gamma : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow (x_1,\ x_5 + x_3,\ x_3,\ x_6 + x_3,\ x_2 + x_3,\ x_4 + x_3)^T$$

$$\delta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow (x_4 + x_6,\ x_3,\ x_2,\ x_1 + x_6,\ x_5,\ x_6)^T$$

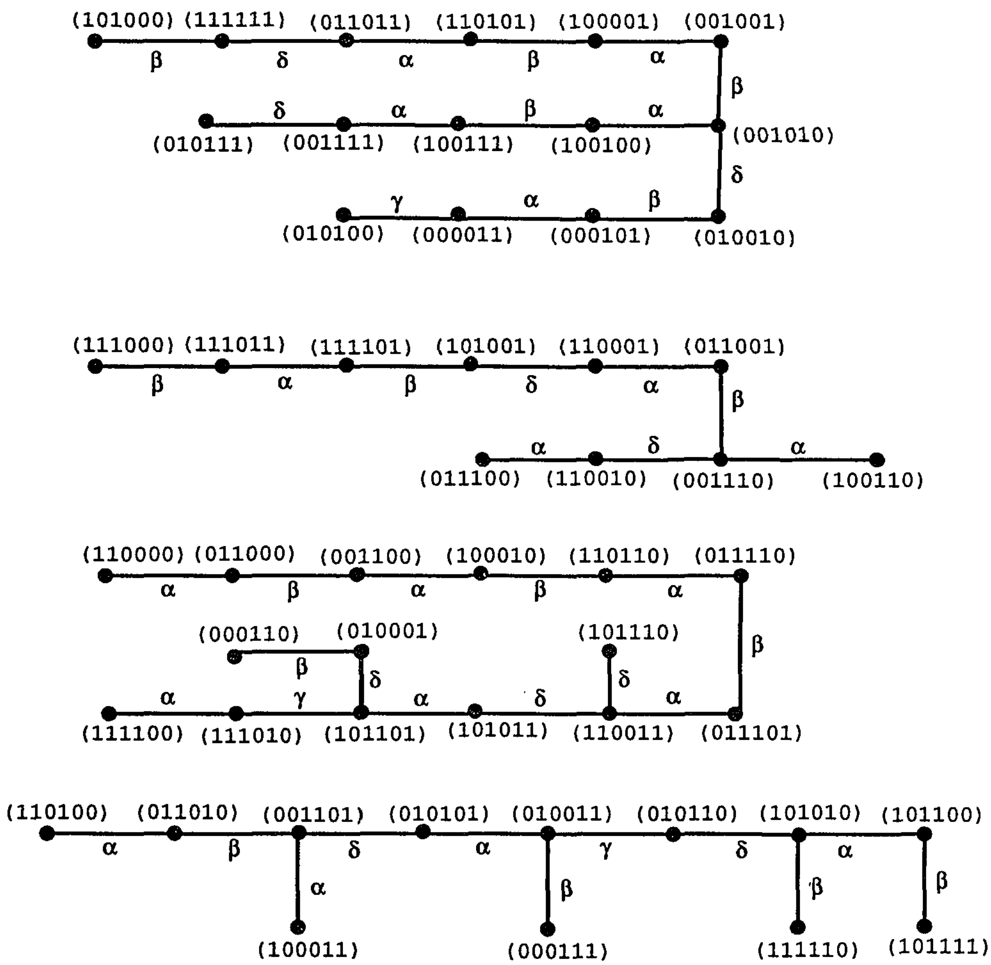


**Figure A-9**

**The matroid $\mathbf{T_{12}/e}$**

$$\begin{array}{cc}
 & \begin{array}{cccccc} 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
1\ 2\ 3\ 4\ 5 & \\
\left(\begin{array}{c} \\ \\ I_5 \\ \\ \\ \end{array}\right. & \left|\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1
\end{array}\right)
\end{array}$$

**Table 10a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $(T_{12}/e, ext1)$ | Circuits | 0 | 7 | 16 | 26 | 21 | | |
| | Cocircuits | 0 | 0 | 4 | 6 | 8 | 8 | 2 |
| $(T_{12}/e, ext2)$ | Circuits | 0 | 8 | 15 | 23 | 20 | | |
| | Cocircuits | 0 | 0 | 5 | 5 | 6 | 10 | 2 |
| $(T_{12}/e, ext3)$ | Circuits | 0 | 6 | 17 | 29 | 20 | | |
| | Cocircuits | 0 | 0 | 3 | 7 | 10 | 6 | 2 |
| $(T_{12}/e, ext4)$ | Circuits | 0 | 7 | 16 | 26 | 19 | | |
| | Cocircuits | 0 | 0 | 4 | 6 | 8 | 8 | 2 |

**Table 10b. Single-element extensions of $\mathbf{T_{12}/e}$**

| $(T_{12}/e, ext1)$ | $(T_{12}/e, ext2)$ | $(T_{12}/e, ext3)$ | $(T_{12}/e, ext4)$ |
|---|---|---|---|
| (10100) | **(11100)** | (10010) | (11110) |
| (01010) | (00110) | (10110) | (01001) |
| **(11010)** | (10001) | **(10101)** | (11001) |
| (10011) | (00101) | (01011) | (01101) |
| (10111) | (01111) | (11011) | **(00011)** |

**Automorphisms of $\mathbf{T_{12}/e}$**

Pivoting on element $[a_{4,11}]$ and swapping row 1 with row 3 induces an automorphism on $T_{12}/e$. Pivoting on the element $[a_{2,6}]$ and then on the element $[a_{3,8}]$ also induces an automorphism on $T_{12}/e$. The corresponding maps on $(x_1,\ x_2,\ x_3\ ,x_4,\ x_5)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3\ ,x_4,\ x_5)^T \longrightarrow (x_3 + x_4,\ x_2 + x_4,\ x_1 + x_4,\ x_4,\ x_5 + x_4)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5)^T \longrightarrow (x_1 + x_2,\ x_2,\ x_3,\ x_4 + x_3,\ x_5 + x_3)^T$$

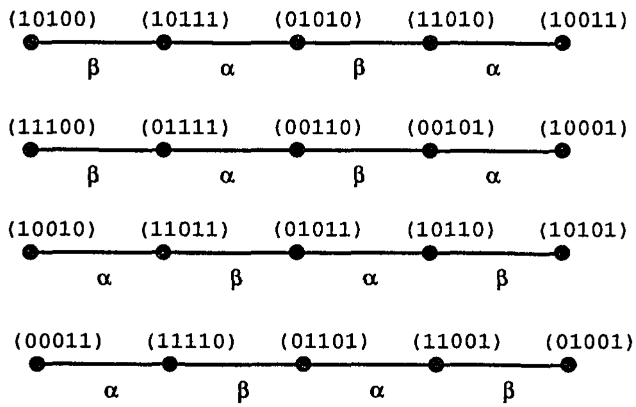


**Figure A-10**

**The matroid $T_{12}\backslash e$**

$$
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\left(\begin{array}{c|ccccc}
 & 1 & 1 & 0 & 0 & 0 \\
 & 0 & 1 & 1 & 0 & 0 \\
 & 0 & 0 & 1 & 1 & 1 \\
I_6 & 1 & 1 & 1 & 0 & 1 \\
 & 0 & 1 & 1 & 1 & 0 \\
 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
\end{array}
$$

**Table 11a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(T_{12}\backslash e, ext1)$ | Circuits | 0 | 1 | 10 | 12 | 16 | 10 |
| | Cocircuits | 0 | 1 | 10 | 12 | 16 | 10 |
| $(T_{12}\backslash e, ext2)$ | Circuits | 0 | 3 | 10 | 7 | 16 | 6 |
| | Cocircuits | 0 | 4 | 6 | 11 | 13 | 9 |
| $(T_{12}\backslash e, ext3)$ | Circuits | 0 | 1 | 10 | 13 | 16 | 8 |
| | Cocircuits | 0 | 2 | 6 | 17 | 15 | 7 |
| $(T_{12}\backslash e, ext4)$ | Circuits | 0 | 0 | 16 | 0 | 30 | 0 |
| | Cocircuits | 0 | 2 | 7 | 14 | 16 | 8 |
| $(T_{12}\backslash e, ext5)$ | Circuits | 0 | 2 | 10 | 9 | 16 | 9 |
| | Cocircuits | 0 | 2 | 10 | 9 | 16 | 9 |
| $(T_{12}\backslash e, ext6)$ | Circuits | 0 | 2 | 10 | 10 | 16 | 8 |
| | Cocircuits | 0 | 3 | 6 | 14 | 15 | 6 |
| $(T_{12}\backslash e, ext7)$ | Circuits | 0 | 0 | 15 | 0 | 33 | 0 |
| | Cocircuits | 0 | 1 | 9 | 15 | 12 | 12 |
| $(T_{12}\backslash e, ext8)$ | Circuits | 0 | 2 | 10 | 10 | 16 | 6 |
| | Cocircuits | 0 | 3 | 6 | 14 | 14 | 8 |

**Table 11a. Sizes of circuits and cocircuits (continued)**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $(T_{12}\backslash e, ext9)$ | Circuits | 0 | 0 | 10 | 15 | 16 | 11 |
| | Cocircuits | 0 | 0 | 10 | 15 | 16 | 11 |
| $(T_{12}\backslash e, ext10)$ | Circuits | 0 | 0 | 17 | 0 | 27 | 0 |
| | Cocircuits | 0 | 3 | 5 | 13 | 18 | 8 |
| $(T_{12}\backslash e, ext11)$ | Circuits | 0 | 0 | 16 | 0 | 30 | 0 |
| | Cocircuits | 0 | 2 | 7 | 14 | 15 | 10 |
| $(T_{12}\backslash e, ext12)$ | Circuits | 0 | 0 | 15 | 0 | 32 | 0 |
| | Cocircuits | 0 | 0 | 15 | 0 | 32 | 0 |

**Table 11b. Single-element extensions of $\mathbf{T_{12}\backslash e}$**

| $(T_{12}\backslash e, ext1)$ | $(T_{12}\backslash e, ext2)$ | $(T_{12}\backslash e, ext3)$ | $(T_{12}\backslash e, ext4)$ |
|---|---|---|---|
| (110000) | (101000) | (011000) | (111000) |
| (111100) | (010010) | (100010) | (110010) |
| (110110) | (001001) | (101110) | (001110) |
| (010001) | (000101) | (011110) | (101001) |
| (101011) | (001111) | (110011) | (011001) |

| $(T_{12}\backslash e, ext5)$ | $(T_{12}\backslash e, ext6)$ | $(\mathbf{T_{12}\backslash e, ext7})$ | $(T_{12}\backslash e, ext8)$ |
|---|---|---|---|
| (100100) | (001100) | (011100) | (001010) |
| (010100) | (111010) | (100110) | (100001) |
| (110101) | (000110) | (110001) | (011011) |
| (000011) | (101101) | (111101) | (100111) |
| (111111) | (011101) | (111011) | (010111) |

**Table 11b. Single-element extensions of $T_{12}\backslash e$ (continued)**

| $(T_{12}\backslash e, ext9)$ | $(T_{12}\backslash e, ext10)$ | $(T_{12}\backslash e, ext11)$ | $(T_{12}\backslash e, ext12)$ |
|---|---|---|---|
| (111001) | **(101100)** | **(111110)** | (110100) |
| | (101010) | (010101) | |
| | (011010) | (100011) | |
| | (010110) | (010011) | |
| | (000111) | (101111) | |

## Automorphisms of $\mathbf{T_{12}}\backslash$e

Pivoting on element $[a_{1,8}]$, then on element $[a_{3,9}]$, and swapping row 2 with row 6 and row 4 with row 5 induces an automorphism on $T_{12}\backslash e$. Pivoting on element $[a_{4,7}]$, then on element $[a_{5,9}]$, and swapping row 1 with row 6 induces an automorphism on $T_{12}\backslash e$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T$ are shown below:

$\alpha : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow$

$(x_1,\ x_6 + x_1 + x_3,\ x_3,\ x_5 + x_1 + x_3,\ x_4 + x_1 + x_3,\ x_2 + x_1 + x_3)^T$

$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow (x_6 + x_4,\ x_2 + x_5,\ x_3 + x_5,\ x_4 + x_5,\ x_5,\ x_1 + x_4 + x_5)^T$

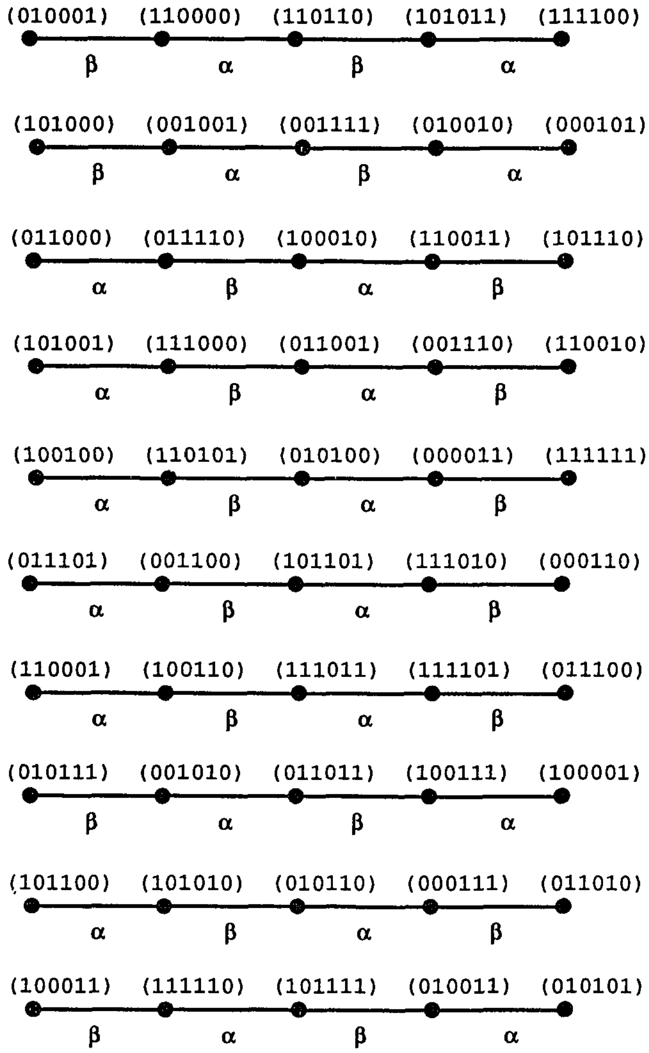
(010001) (110000) (110110) (101011) (111100)
β α β α
(101000) (001001) (001111) (010010) (000101)
β α β α
(011000) (011110) (100010) (110011) (101110)
α β α β
(101001) (111000) (011001) (001110) (110010)
α β α β
(100100) (110101) (010100) (000011) (111111)
α β α β
(011101) (001100) (101101) (111010) (000110)
α β α β
(110001) (100110) (111011) (111101) (011100)
α β α β
(010111) (001010) (011011) (100111) (100001)
β α β α
(101100) (101010) (010110) (000111) (011010)
α β α β
(100011) (111110) (101111) (010011) (010101)
β α β α

**Figure A-11**

**The matroid $T_{12}$**

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$
$$\left(\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 0 \\ I_6 & 1 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

**Table 12a. Sizes of circuits and cocircuits**

| | Size | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $(T_{12}, ext1)$ | Circuits | 0 | 2 | 15 | 16 | 32 | 16 | |
| | Cocircuits | 0 | 0 | 7 | 8 | 16 | 16 | 4 |
| $(T_{12}, ext2)$ | Circuits | 0 | 3 | 15 | 12 | 32 | 15 | |
| | Cocircuits | 0 | 0 | 9 | 6 | 12 | 20 | 3 |
| $(T_{12}, ext3)$ | Circuits | 0 | 1 | 15 | 20 | 32 | 17 | |
| | Cocircuits | 0 | 0 | 5 | 10 | 20 | 12 | 5 |
| $(T_{12}, ext4)$ | Circuits | 0 | 0 | 23 | 0 | 56 | 0 | |
| | Cocircuits | 0 | 0 | 6 | 9 | 16 | 16 | 6 |

**Table 12b. Single-element extensions of $\mathbf{T_{12}}$**

| $(T_{12}, ext1)$ | $(T_{12}, ext2)$ | $(T_{12}, ext3)$ | $(T_{12}, ext4)$ |
|---|---|---|---|
| (110000) | (101000) | (011000) | (111000) |
| (001100) | (100100) | (100010) | (101100) |
| (111100) | (010100) | (101110) | (011100) |
| (001010) | (010010) | (011110) | (110010) |
| (111010) | (001001) | (111001) | (101010) |
| (000110) | (000101) | (110011) | (011010) |
| (110110) | (110101) | | (100110) |
| (100001) | (000011) | | (010110) |
| (010001) | (001111) | | (001110) |
| (101101) | (111111) | | (111110) |
| (011101) | | | (110001) |
| (101011) | | | (101001) |
| (011011) | | | (011001) |
| (100111) | | | (010101) |
| (010111) | | | (111101) |
| | | | (100011) |
| | | | (010011) |
| | | | (111011) |
| | | | (000111) |
| | | | (101111) |

**Automorphisms of $\mathbf{T_{12}}$**

Swapping row 1 with row 3, row 2 with row 5, and row 4 with row 6 induces an automorphism on $T_{12}$. Pivoting on element $[a_{1,8}]$ and swapping row 4 with row 5 also induces an automorphism on $T_{12}$. Pivoting on element $[a_{3,9}]$ and swapping row 2 with row 6, induces an automorphism on $T_{12}$. Pivoting on element $[a_{4,12}]$ and swapping row 1 with row 2 and row 3 with row 5, induces an automorphism on $T_{12}$. The corresponding maps on $(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T$ are shown below:

$$\alpha : (x_1,\ x_2,\ x_3\ ,x_4,\ x_5,\ x_6)^T \longrightarrow (x_3,\ x_5,\ x_1,\ x_6,\ x_2,\ x_4)^T$$

$$\beta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow (x_1,\ x_2 + x_1,\ x_3,\ x_5 + x_1,\ x_4 + x_1, x_6 + x_1)^T$$

$$\gamma : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow (x_1,\ x_6 + x_3,\ x_3,\ x_4 + x_3,\ x_5 + x_3, x_2 + x_3)^T$$

$$\delta : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)^T \longrightarrow (x_2 + x_4,\ x_1 + x_4,\ x_5,\ x_4,\ x_3, x_6)^T$$

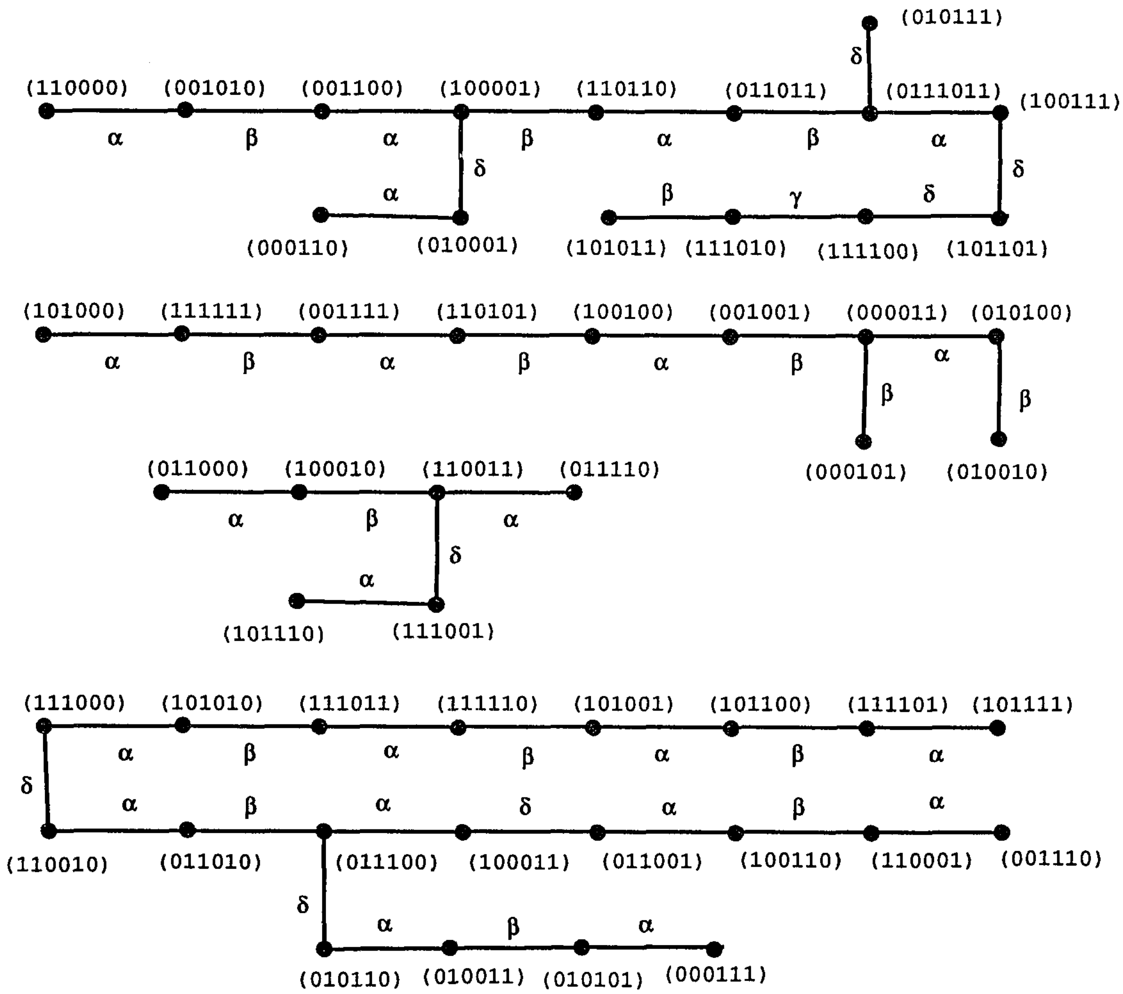

**Figure A-12**

**Table 12c. Circuits of $T_{12}$**

| | | | | |
|---|---|---|---|---|
| 1 2 6 7 | 1 2 3 12 | 1 3 5 8 9 10 | 1 4 7 9 10 11 | 2 5 8 9 10 12 |
| 1 5 6 8 | 3 6 7 12 | 1 3 4 6 7 11 | 2 3 5 6 8 12 | 2 4 6 7 11 12 |
| 2 5 7 8 | 7 9 10 12 | 3 4 5 7 8 11 | 1 3 5 7 8 12 | 4 5 6 8 11 12 |
| 4 5 6 9 | 1 4 11 12 | 2 3 5 6 9 11 | 3 4 5 7 9 12 | 1 5 6 9 11 12 |
| 1 4 8 9 | 8 9 11 12 | 1 3 5 7 9 11 | 2 3 4 8 9 12 | 2 5 7 9 11 12 |
| 3 4 5 10 | 1 2 4 5 7 9 | 1 2 3 8 9 11 | 1 2 4 5 10 12 | 1 3 5 10 11 12 |
| 3 6 9 10 | 2 4 6 7 8 9 | 3 6 7 8 9 11 | 4 5 6 7 10 12 | 3 6 8 10 11 12 |
| 2 3 4 11 | 1 3 4 6 8 10 | 1 5 6 7 10 11 | 2 4 6 8 10 12 | |
| 2 5 10 11 | 2 3 4 7 8 10 | 1 2 6 8 10 11 | 1 4 7 8 10 12 | |
| 7 8 10 11 | 1 2 3 7 9 10 | 2 4 6 9 10 11 | 1 2 6 9 10 12 | |

# VITA

Sandra Reuben Kingan was born in Bombay, India on December 5, 1966. She graduated from St. Xavier's College in June 1987 with a Bachelor of Science degree in Mathematics. She received a Master of Science degree in Mathematics from LSU in May, 1990, and is presently a candidate for the doctoral degree in Mathematics at LSU.

# DOCTORAL EXAMINATION AND DISSERTATION REPORT

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**Major Field:** Mathematics

**Title of Dissertation:** Structural Results for Matroids

**Approved:**

James G Oxley

Major Professor and Chairman

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**Date of Examination:**

May 6, 1994